$$C_n = \bigcup_{j=0}^{n} (\text{closure } R_j) \cup \{\mathcal{S} \sim \bigcup_{j=0}^{\infty} (\text{closure } R_j)\}.$$

Clearly, for each integer n,

$$\bigcup_{j=0}^{\infty} C_j = \mathcal{S}. \qquad C_n \subset C_{n+1} \in F,$$

After checking the hereditariness of F, we infer from 4.2 that each open set is ϕ measurable F. Hence, if we recall 3.5, C_n is ϕ measurable F for each integer n. Thus F is ϕ convenient. Reference to 4.3 completes the proof.

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ON THE DISTRIBUTION OF THE VALUES OF $|f(z)|$ IN THE UNIT CIRCLE

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1. **Summary.** Let $f(z)=1+a_1z+\cdots$ be analytic for $|z|\leqq 1$, $f(z)\not\equiv 1$. Then $|f(z)|$ will be greater than 1 at some points of the unit circle, and less than 1 at others. Calling $A(f)$ the area of the set of points within the unit circle, for which $|f(z)|\geqq 1$, let α and β be the two largest non-negative constants such that $\alpha\leqq A(f)\leqq\pi-\beta$, for every $f(z)$. It is shown that $\alpha=\beta=0$; in other words, if ϵ is arbitrarily small positive, there are functions $f(z)$ such that $A(f)<\epsilon$, and others such that $A(f)>\pi-\epsilon$. The same is true, if $f(z)$ is restricted to polynomials $\prod_{\nu=1}^{n}(z-z_\nu)$ with $\prod_{\nu=1}^{n}|z_\nu|=1$. These statements will be proved in §2. §3 contains a few additional results, given without proofs.

2. **Proofs.** The statements made in the summary are contained in the following theorem.

THEOREM. *Let P stand for the set of polynomials over the complex field of the form $f(z)=\prod_{\nu=1}^{n}(z-z_\nu)$, with $\prod_{\nu=1}^{n}|z_\nu|=1$; let $A(f)$ denote the area of the set of points in the unit circle, for which $|f(z)|\geqq 1$; let ϵ be an arbitrarily small positive number. Then P contains polynomials $f_1(z)$ and $f_2(z)$ such that $A(f_1)>\pi-\epsilon$, and $A(f_2)<\epsilon$.*

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PROOF. Consider $f_1(z)=1+Nz+z^2$, where N is a real positive number greater than 3. Then in the unit circle, $|f(z)| \geqq |Nz|-2$, which is greater than 1, if $|Nz|>3$, $|z|>3/N$. Thus

$$A(f) > \pi - \frac{9}{N^2}\pi,$$

and for $N>3(\pi/\epsilon)^{1/2}$, this is greater than $\pi-\epsilon$. This proves the first part of the theorem.

To prove the second part, consider a function $F_0(z)=b_1z+b_2z^2+\cdots$, analytic for $|z|\leqq 1$, $F(z)\not\equiv 0$. The real and the imaginary part of a function $F(z)$ will be designated by Re $F(z)$ and Im $F(z)$, respectively. Call $\overline{B}(F_0)$ the set of points in the unit circle for which Re $F_0(z)\geqq 0$, and $B(F_0)$ the area of $\overline{B}(F_0)$. Let n_1 be a positive real number.

The function $F_1(z)=\exp\,(n_1F_0(z))-1=n_1b_1z+\cdots$ is again analytic for $|z|\leqq 1$. Its real part is Re $F_1(z)=\exp\,(n_1\text{Re } F_0(z))\cdot\cos\,(n_1\text{Im}F_0(z))-1$. Therefore Re $F_1(z)$ is positive or zero only where Re $F_0(z)\geqq 0$ *and* $\cos\,(n_1\text{Im } F_0(z))\geqq\exp\,(-n_1\text{Re } F_0(z))>0$. It will be shown (see the lemma below), that for sufficiently large values of n_1, $\cos\,(n_1 \text{ Im } F_0(z))$ is negative in a subregion of $\overline{B}(F_0)$ of area greater than $(1/3)B(F_0)$. Then $B(F_1)<(2/3)B(F_0)<(2/3)\pi$. Similarly a function $F_2(z)=\exp\,(n_2F_1(z))-1$ may be formed such that $B(F_2)<(2/3)B(F_1)<(2/3)^2\pi$. Continuing in this way, for any arbitrarily chosen positive number ϵ, a function $F_t(z)=c_1z+\cdots$ may be formed which is analytic for $|z|\leqq 1$, and for which $B(F_t)<(2/3)^t\pi<\epsilon/3$, for t large enough.

Then the function $f(z)=\exp\,(F_t(z))=1+a_1z+\cdots$ is analytic for $|z|\leqq 1$, and $|f(z)|=\exp\,(\text{Re } F_t(z))$. Therefore $|f(z)|\geqq 1$ for the same points for which Re $F_t(z)\geqq 0$. If $A(f)$ stands again for the area of the set of points in the unit circle for which $|f(z)|\geqq 1$, $A(f)=B(F_t)<\epsilon/3$.

Now choose a positive number η so small that the set of points in the unit circle where $1-2\eta\leqq|f(z)|\leqq 1$ has an area not exceeding $\epsilon/3$. Finally choose a positive integer q so large that $|a_qz^q+a_{q+1}z^{q+1}+\cdots|<\eta$ for $|z|\leqq 1$ and that $|z^q|<\eta$ for $|z|\leqq 1-\epsilon/(6\pi)$. Then in the unit circle $|1+a_1z+\cdots+a_{q-1}z^{q-1}+z^q|<1-2\eta+\eta+\eta=1$, except in the two regions where either $|f(z)|\geqq 1-2\eta$, or $|z|>1-\epsilon/(6\pi)$. The area of the first region is less than $\epsilon/3+\epsilon/3=2\epsilon/3$, and the area of the second one is less than $2\pi\cdot\epsilon/(6\pi)=\epsilon/3$. Thus $A(f_2)<\epsilon$, with $f_2(z)=1+a_1z+\cdots+a_{q-1}z^{q-1}+z^q$.

In the proof of the theorem, use has been made of the following lemma.

LEMMA. *If* $F(z)=b_1z+b_2z^2+\cdots\not\equiv 0$ *is analytic for* $|z|\leqq 1$, *if* $\overline{B}$ *is the set of points of the unit circle for which* Re $F(z)\geqq 0$, *and* B *the area of* $\overline{B}$, *and if* n *is a sufficiently large positive number, then* cos (n Im $F(z)$) *is negative in a subset of* $\overline{B}$ *whose area is greater than* $B/3$.

PROOF OF THE LEMMA. For $z=re^{i\theta}$, Im $F(z)$ is a function $g_r(\theta)$ of r and θ. Choose a positive number m so small that the absolute value of $g_r'(\theta)=\partial/\partial\theta(\text{Im } F(z))$ will be not less than m except in a subset of $\overline{B}$ whose area is less than $B/12$. In other words, if $\overline{B}'$ is the set of points in the unit circle where Re $F(z)\geqq 0$ *and* $|g_r'(\theta)|\geqq m$, then B', the area of $\overline{B}'$, will be between $11B/12$ and B.

Call M the largest value of $|g_r'(\theta)|$ in $|z|\leqq 1$, T the largest value of $|g_r''(\theta)|$ in $|z|\leqq 1$, and S the largest number of intersections between the boundary of $\overline{B}'$ and any of the circles $r=\text{constant}\leqq 1$. For every r, this number of intersections is not greater than the sum of the numbers of intersections between $r=\text{const.}$ and the three curves Re $F(z)=0$; $\partial/\partial\theta$ (Im $F(z)$) $=+m$; $\partial/\partial\theta$(Im $F(z)$) $=-m$. Thus a finite value of S certainly exists. Therefore we know that, for every r,

$$(1)\qquad m\leqq |g_r'(\theta)|\leqq M,\quad\text{and}\quad |g_r''(\theta)|\leqq T,\text{ in }\overline{B}'.$$

(2) $\overline{B}'$ contains less than S separate intervals of the circumference of the circle $r=\text{constant}$.

The subscript r in $g_r(\theta)$ will be omitted from here on.

Let (σ,τ) $(\sigma\leqq\theta\leqq\tau)$ be one of the intervals mentioned in (2). Then $g'(\theta)$ is either not less than m throughout (σ,τ), or not greater than $-m$ throughout. Assume $g'(\theta)\geqq m$ in (σ,τ). Choose a positive number n, and call $\theta_1,\theta_2,\cdots,\theta_s$ the values of θ in (σ,τ) for which $\cos(ng(\theta))=0$ $(\sigma\leqq\theta_1<\theta_2<\cdots<\theta_s\leqq\tau)$. Assume for the present that $s\geqq 2$. Then

$$(3)\qquad ng(\theta_{i+1})-ng(\theta_i)=\pi,\qquad\text{for } i=1,2,\cdots,s-1.$$

In the subinterval (θ_i,θ_{i+1}), $g(\theta)=g(\theta_i)+(\theta-\theta_i)g'(\bar\theta)$, with $\theta_i\leqq\bar\theta\leqq\theta$. Since $g'(\bar\theta)$ differs from $(g(\theta_{i+1})-g(\theta_i))/(\theta_{i+1}-\theta_i)$ absolutely by less than $(\theta_{i+1}-\theta_i)\cdot\max g''(\theta)$,

$$g(\theta)=g(\theta_i)+(\theta-\theta_i)\frac{g(\theta_{i+1})-g(\theta_i)}{\theta_{i+1}-\theta_i}+\kappa(\theta_{i+1}-\theta_i)^2T,\text{ with }|\kappa|<1,$$

or, using (3),

$$ng(\theta)=ng(\theta_i)+\frac{\pi(\theta-\theta_i)}{\theta_{i+1}-\theta_i}+n\kappa(\theta_{i+1}-\theta_i)^2T.$$

Since $\cos(ng(\theta_i))=0$, $ng(\theta_i)=2q\pi\mp\pi/2$ (q=integer). Here the upper or the lower sign applies according to whether $\cos(ng(\theta))$ is positive or negative throughout (θ_i, θ_{i+1}). Thus

$$\cos(ng(\theta)) = \cos\left[\mp\frac{\pi}{2}+\frac{\pi(\theta-\theta_i)}{\theta_{i+1}-\theta_i}+\kappa n(\theta_{i+1}-\theta_i)^2T\right]$$

$$= \pm\sin\left[\frac{\pi(\theta-\theta_i)}{\theta_{i+1}-\theta_i}+\kappa n(\theta_{i+1}-\theta_i)^2T\right]$$

$$= \pm\sin\frac{\pi(\theta-\theta_i)}{\theta_{i+1}-\theta_i}+\kappa' n(\theta_{i+1}-\theta_i)^2T \text{ with } |\kappa'|\leqq|\kappa|<1.$$

Therefore

$$(4)\quad \int_{\theta_i}^{\theta_{i+1}}\cos(ng(\theta))d\theta = \pm\frac{2}{\pi}(\theta_{i+1}-\theta_i)+\lambda_i n(\theta_{i+1}-\theta_i)^3T, \quad \text{with } |\lambda_i|<1.$$

For any θ-interval (γ, δ), let $I_{\gamma\delta}$ stand for the sum of the lengths of all those subintervals of (γ, δ), where $\cos(ng(\theta))$ is negative. Then $(\delta-\gamma)-I_{\gamma\delta}$ is equal to the sum of the lengths of the subintervals of (γ, δ), where $\cos(ng(\theta))$ is positive.

From (4) follows

$$\int_{\theta_1}^{\theta_s}\cos(ng(\theta))d\theta = \frac{2}{\pi}\left[(\theta_s-\theta_1-I_{\theta_1\theta_s})-I_{\theta_1\theta_s}\right] + \lambda(s-1)nT\cdot\max(\theta_{i+1}-\theta_i)^3 \quad \text{with } |\lambda|<1.$$

According to (1), $m\leqq(g(\theta_{i+1})-g(\theta_i))/(\theta_{i+1}-\theta_i)\leqq M$. Therefore, and because of (3),

$$(5)\quad \frac{\pi}{nM}\leqq\theta_{i+1}-\theta_i\leqq\frac{\pi}{nm} \quad \text{for } i=1, 2, \cdots, s-1, \text{ and } s-1\leqq\frac{\tau-\sigma}{\pi/nM}<2nM.$$

Similarly, since $ng(\theta_1)-ng(\sigma)<\pi$, and $ng(\tau)-ng(\theta_s)<\pi$:

$$(5')\quad \theta_1-\sigma<\frac{\pi}{nm}, \quad \text{and} \quad \tau-\theta_s<\frac{\pi}{nm}\cdot$$

Thus $|\lambda(s-1)nT\max(\theta_{i+1}-\theta_i)^3|<2nMnT\pi^3/(nm)^3=2\pi^3MT/(nm^3)$, and

$$\int_{\theta_1}^{\theta_s} \cos (ng(\theta))d\theta = \frac{4}{\pi}\left[\frac{\theta_s - \theta_1}{2} - I_{\theta_1\theta_s}\right] + \lambda' \frac{2\pi^3 MT}{nm^3},$$

with $|\lambda'| < 1$.

Therefore $(4/\pi)[(\tau-\sigma)/2 - I_{\sigma\tau}]$ differs from $\int_{\theta_1}^{\theta_s} \cos (ng(\theta))d\theta$ absolutely by less than $(2/\pi)(\theta_1-\sigma+\tau-\theta_s)+2\pi^3 MT/(nm^3)$, or finally by (5′)

$$\left|\int_{\theta_1}^{\theta_s} \cos (ng(\theta))d\theta - \frac{4}{\pi}\left[\frac{\tau - \sigma}{2} - I_{\sigma\tau}\right]\right| < \frac{C_1}{n}, \tag{6}$$

where $C_1 = 4/m + 2\pi^3 MT/m^3$ is independent of r and n.

On the other hand, if $s \geqq 3$, for $i = 2, 3, \cdots, s-1$:

$$\int_{\theta_{i-1}}^{\theta_{i+1}} \cos (ng(\theta))d\theta = \int_{g(\theta_{i-1})}^{g(\theta_{i+1})} \cos (ng) \frac{d\theta}{dg} dg,$$

$$\frac{d\theta}{dg} = \left(\frac{d\theta}{dg}\right)_{\theta_i} + (g(\theta) - g(\theta_i))\left(\frac{d^2\theta}{dg^2}\right)_{\bar{\theta}},$$

with $\bar{\theta}$ between θ_i and θ.

By (1), $|d^2\theta/dg^2| = |g''(\theta)/(g'(\theta))^3| \leqq T/m^3$, and by (3), for $\theta_{i-1} \leqq \theta \leqq \theta_{i+1}$, $|g(\theta) - g(\theta_i)| \leqq \pi/n$. Therefore

$$\int_{\theta_{i-1}}^{\theta_{i+1}} \cos (ng(\theta))d\theta = \left(\frac{d\theta}{dg}\right)_{\theta_i} \cdot \int_{g(\theta_{i-1})}^{g(\theta_{i+1})} \cos (ng)dg + \mu 2 \frac{\pi^2}{n^2}\frac{T}{m^3},$$

with $|\mu| < 1$.

Since $ng(\theta_{i-1})$ and $ng(\theta_{i+1})$ differ by 2π,

$$\int_{g(\theta_{i-1})}^{g(\theta_{i+1})} \cos (ng)dg = 0, \quad \text{and} \quad \left|\int_{\theta_{i-1}}^{\theta_{i+1}} \cos (ng(\theta))d\theta\right| < \frac{2\pi^2 T}{n^2 m^3}.$$

Thus

$$\begin{aligned} \left|\int_{\theta_1}^{\theta_s} \cos (ng(\theta))d\theta\right| &< (\theta_s - \theta_{s-1}) + \frac{s-1}{2}\frac{2\pi^2 T}{n^2 m^3} \\ &< \frac{\pi}{nm} + \frac{2\pi^2 MT}{nm^3} = \frac{C_2}{n}, \end{aligned} \tag{7}$$

according to (5). Formula (7) is obviously also correct if $s = 2$. The term $\theta_s - \theta_{s-1}$ may be omitted if s is odd. The constant $C_2 = \pi/m + 2\pi^2 MT/m^3$ is again independent of r and n.

Combining (6) and (7) we get

$$(8)\qquad \left|\frac{\tau-\sigma}{2}-I_{\sigma\tau}\right|<\frac{C}{n},\quad \text{with}\quad C=\frac{\pi}{4}(C_1+C_2).$$

In the derivation of (8) it has been assumed that $s\geqq 2$, that there are at least two values of θ in (σ, τ) for which $\cos(ng(\theta))=0$. If there is only one such value in (σ, τ), or none at all, then by (5′)

$$\left|\frac{\tau-\sigma}{2}-I_{\sigma\tau}\right|\leqq\frac{1}{2}(\tau-\sigma)<\frac{1}{2}2\frac{\pi}{mn}<\frac{C}{n},\quad \text{since}\quad \frac{\pi}{m}<\frac{\pi}{4}C_1<C.$$

If $g'(\theta)\leqq -m$ throughout (σ, τ), the same conclusion is reached in a similar manner.

Thus (8) is correct for any interval (σ, τ).

From (2) and (8) follows: for any value of r between 0 and 1, the sum of the lengths of *all* the θ-intervals within $\overline{B}'$ where $\cos(n \operatorname{Im} F(z))$ is negative differs from half the sum of the total lengths of the θ-intervals within $\overline{B}'$ by less than SC/n. Therefore the area of the subset of $\overline{B}'$ formed by the points for which $\cos(n\operatorname{Im} F(z))$ is negative differs from $B'/2$ by less than $(SC/n)\cdot\int_0^1 r\,dr=SC/(2n)$. Take n so large that $SC/(2n)<B/12$. Then, since $B'>11B/12$, $\cos(n\operatorname{Im} F(z))$ is negative in a subset of $\overline{B}'$ (and therefore of $\overline{B}$) whose area is greater than $11B/24-B/12>B/3$. This completes the proof of the lemma.

3. **Some additional remarks.** Let $f(z)=\prod_{\nu=1}^{n}(z-z_\nu)$ be further restricted by the condition that $|z_\nu|=1$ for $\nu=1, 2, \cdots, n$. Let α and β again be the two largest possible numbers such that $\alpha\leqq A(f)\leqq\pi-\beta$ for every $f(z)$ of this form. In this case the values of α and β are still unknown. It can be shown that $\alpha=\beta$, and that $\alpha\leqq .43$. Dr. Erdös[1] quotes Mr. Eröd as possessing an unpublished proof that $\alpha>0$.

It is possible, however, to construct polynomials $f(z)=\prod_{\nu=1}^{n}(z-z_\nu)$ with $\prod_{\nu=1}^{n}|z_\nu|=1$ such that $A(f)<\epsilon_1$, and $1-\epsilon_2\leqq|z_\nu|\leqq 1+\epsilon_2$ for $\nu=1, 2, \cdots, n$, where ϵ_1 and ϵ_2 are arbitrarily small positive numbers which are independent of each other.

If $F(z)=b_1z+\cdots$ is analytic for $|z|\leqq 1$ and such that the set $B(F)$ and its complement in the unit circle are both simply connected regions, then it can be proved that there exist positive numbers α such that for every $F(z)$ of this kind $\alpha\leqq B(F)\leqq\pi-\alpha$. The largest possible value of α in this case can be shown to be not less than .141 nor greater than .283.

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[1] Paul Erdös, *Note on some elementary properties of polynomials*, Bull. Amer. Math. Soc. vol. 46 (1940) p. 954.