CORRECTION TO MY PAPER "NOTE ON AFFINELY CONNECTED MANIFOLDS"

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In my recent paper [1][1] *Note on affinely connected manifolds* I gave a proof of a theorem on affinely connected manifolds. As was pointed out by H. Whitney, the fact that I made use of in the proof, that the space of all real matrices (a_i^j), $|a_i^j|>0$, is simply connected, is erroneous. But the theorem I intended to prove is true. I present in the following lines a revised proof, in which are clarified at the same time certain ambiguities of that note.

We begin by considering the space $M(n)$ of all real matrices (a_i^j), $i, j=1, \cdots, n$ with $\Delta \equiv |a_i^j|>0$. Let $R(n)$ denote the group space of the proper orthogonal group in n variables. There is a natural way to imbed $R(n)$ in $M(n)$ and it is well known that $R(n)$ is a deformation retract of $M(n)$ [2]. In particular, it follows that $R(n)$ and $M(n)$ have the same homotopy type and hence isomorphic homotopy groups. Thus the fundamental group of $M(n)$ is free cyclic if $n=2$ and is cyclic of order two if $n \geqq 3$.

We denote by $\psi: R(n) \rightarrow M(n)$ the identity mapping and by $f: M(n) \rightarrow R(n)$ the deformation such that under f every point of $R(n)$ remains fixed. Let ψ and f denote at the same time the induced homomorphisms of the (singular) chains and ψ^* and f^* the corresponding dual homomorphisms of the cochains. Since f is a deformation, we have, for every one-dimensional cycle Z of $M(n)$, $Z \sim \psi f(Z)$. It follows that

$$\int_Z \frac{d\Delta}{\Delta} = \int_{\psi f(Z)} \frac{d\Delta}{\Delta} = \int_{f(Z)} \psi^*\left(\frac{d\Delta}{\Delta}\right) = 0,$$

since $\Delta=1$ in $R(n)$. In other words, in $M(n)$ the integral of $d\Delta/\Delta$ over any one-dimensional cycle is zero.

It is possible to express the differential form $d\Delta/\Delta$ in terms of a_i^j. In fact, let b_j^k be defined by $a_i^j b_j^k = b_i^j a_j^k = \delta_i^k$. Then it is easy to verify that

$$d\Delta/\Delta = da_i^j b_j^i.$$

These remarks on the group manifold being made, let us return to the affinely connected manifold M. Let $\mathfrak{F}$ be the vector bundle of all ordered sets of n linearly independent contravariant vectors through

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[1] Numbers in brackets refer to the references cited at the end of the paper.

a point of M, and let π be the projection of $\mathfrak{F}$ onto M, mapping such a set of vectors into the common origin. Then each fibre of $\mathfrak{F}$ is homeomorphic to $M(n)$. Denoting by π^* the dual homomorphism of π, it follows that a cocycle Φ of M has an inverse image $\pi^*\Phi$ in $\mathfrak{F}$. This is in particular true of our differential form $P=R_{kl}dx^k dx^l$.

As local coordinates in $\mathfrak{F}$ we take the local coordinates x^i of M and the components a_j^i of the n contravariant vectors, which we call $\mathfrak{a}_j$. The components of the absolute differentials $D\mathfrak{a}_j$, referred to the vectors $\mathfrak{a}_k$ themselves, are intrinsic differential forms in $\mathfrak{F}$, independent of the choice of the local coordinate system. If b_i^j are defined as above, they are found to be

$$\omega_j^k = (da_j^i + a_j^l \Gamma_{lm}^i dx^m) b_i^k.$$

Contracting the two indices, we get

$$\omega_i^i = d\Delta/\Delta + \Gamma_{ik}^i dx^k.$$

From the definition of the affine curvature tensor we easily derive the relation

$$d(\Gamma_{ik}^i dx^k) = P.$$

It follows that, in the vector bundle $\mathfrak{F}$,

$$d\omega_i^i = d(d\Delta/\Delta + \Gamma_{ik}^i dx^k) = \pi^* P.$$

Suppose we now integrate the differential form P over a two-dimensional simplicial cycle, on whose one-dimensional skeleton a continuous field of n independent contravariant vectors is defined. The field can be extended continuously over the cycle, with one possible singularity at an interior point of each simplex. Choosing an arbitrarily small neighborhood about each singularity, the integral of P over the cycle will differ as little as we please from a finite sum of integrals $\int d\Delta/\Delta$ over one-dimensional cycles in the space of n independent contravariant vectors, that is, in the space $M(n)$. But such an integral of $d\Delta/\Delta$ is zero, according to our remarks on $M(n)$. It follows that the integral of P over the cycle is zero, as was to be proved.

References

1. S. Chern, *Note on affinely connected manifolds*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 820–823.

2. E. Stiefel, *Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten*, Comment. Math. Helv. vol. 8 (1936) pp. 322–323.

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