THE STRUCTURE OF MINIMAL SETS

MARIANO GARCIA AND GUSTAV A. HEDLUND

1. Introduction. A minimal set is a topological space X acted on by a topological group T such that the orbit closure of every point in X coincides with X. If G is a relatively dense subgroup of T, the orbit closure under G of a point of X may or may not coincide with X. In this paper the properties of the orbit closures under such a subgroup are studied and several of the possibilities are analyzed. There is displayed an example of a regularly almost periodic point such that not all points in the orbit closure are regularly almost periodic.

2. The decomposition of a minimal set by relatively dense subgroups. Let X be a compact metric space and let T be an additive abelian topological group. Let f be a continuous transformation of the product space $X \times T$ into X. We denote the image of the point $x \times t$ under f by either $f(x, t)$ or $f^t(x)$. We assume that f defines a transformation group in that if $t, s \in T$, $x \in X$, then

$$f^0(x) = x, \tag{1}$$

$$f^{s+t}(x) = f^s(f^t(x)). \tag{2}$$

It is easily shown that for fixed t in T, the transformation defined by $x \rightarrow f^t(x)$ is a homeomorphism of X onto X. With f satisfying the stated conditions we say that T is a *transformation group* acting on X.

The subset Y of X is *invariant* under the subset A of T if $f^a(Y) = Y$ for all a in A. If G is a subgroup of T and Y is a subset of X which is invariant under G, then Y is a topological space, G is a topological group, and f defines a continuous transformation of the product $Y \times G$ onto Y such that (1) and (2) are satisfied for all points y in Y and all element pairs g, h in G. Thus G is a transformation group acting on Y.

The *orbit* of x is the set $f(x, T)$. If A is a subset of T, the *orbit of x under A* is the set $f(x, A)$.

The space X is *minimal* under T if for every x in X, $\bar{f}(x, T) = X$, where $\bar{f}(x, T)$ denotes the closure of the set $f(x, T)$. The subset A of T is *relatively dense* in T if there exists a compact subset C of T such that $T = A + C$. The point x of X is *almost periodic* under T if, corresponding to any neighborhood U of x, there exists a set A, relatively dense in T, such that $f(x, A) \subset U$.

Presented to the Society, December 31, 1947; received by the editors October 18, 1947.

We shall assume throughout that X is minimal under T. It follows from a theorem of Gottschalk [1, Theorem 1, p. 762][1] that each point of X is almost periodic under T.

Let G be a relatively dense subgroup of T. It follows from another theorem of Gottschalk [1, Theorem 4, p. 764] that each point x of X is almost periodic under G. Thus the set $\bar{f}(x, G)$ is invariant and minimal under G. If x and y are arbitrary points of X, the sets $\bar{f}(x, G)$ and $\bar{f}(y, G)$ are either disjoint or identical and G effects a decomposition of X into disjoint closed sets which are minimal under G.

Let X and Y be minimal sets with associated transformation groups T and S, and transformations f and g, respectively. The minimal sets X and Y will be said to be *equivalent* if there exists a homeomorphism h of X onto Y and an isomorphism I between T and S such that if $x\in X$, $t\in T$ and $I(t)=s\in S$, then

$$f^t(x) = h^{-1}g^s[h(x)].$$

In this case it will be said that X and Y are *equivalent by virtue of the homeomorphism h and the isomorphism I.* If the groups T and S happen to be identical, X and Y will be said to be *equivalent by virtue of the homeomorphism h.*

THEOREM 1. *Let X be a compact metric space which is minimal under the transformation group T and let G be a relatively dense subgroup of T. Then if x and y are arbitrary points of X, there exists an element u of T such that $f^u[\bar{f}(x, G)]=\bar{f}(y, G)$ and the sets $\bar{f}(x, G)$ and $\bar{f}(y, G)$, which are minimal under G, are equivalent by virtue of f^u.*

Since G is a relatively dense subgroup of T, there exists a compact set A in T such that $T=A+G$. Let X_1 be the subset of X consisting of all points $f(z, a)$ such that $z\in\bar{f}(x, G)$ and $a\in A$. If follows from the compactness of $\bar{f}(x, G)$ and A that X_1 is compact. But if t is any element of T, then $t=a+g$, with a in A and g in G, and since $f^g(x) \in\bar{f}(x, G)$, we infer that $f^t(x)=f[f^g(x), a]\in X_1$. Since the orbit of x is everywhere dense in X, X_1 is everywhere dense in X and thus, X_1 being a closed set, $X_1=X$. In particular, we have shown that there exists a point z in $\bar{f}(x, G)$ and an element u of T such that $f^u(z)=y$.

The transformation f^u is a homeomorphism of X onto X. If g is any element of G, $f^u[f^g(z)]=f^g[f^u(z)]=f^g(y)$ and since the set $f(x, G)$ is everywhere dense in $\bar{f}(x, G)$, while the set $f(y, G)$ is everywhere dense in $\bar{f}(y, G)$, it follows that $f^u[\bar{f}(x, G)]=\bar{f}(y, G)$. If w is any point

[1] Numbers in brackets refer to the bibliography at the end of the paper.

of $\bar{f}(x, G)$ and g is any element of G, then

$$f^g(w) = f^{-u}[f^g\{f^u(w)\}]$$

and $\bar{f}(x, G)$ and $\bar{f}(y, G)$ are equivalent by virtue of f^u.

THEOREM 2. *The decomposition of X by G is continuous.*

Let g denote the mapping of $X \times G$ onto X defined by f. According to a theorem of Gottschalk [**1**, Theorem 5, p. 765] it is sufficient to prove that g is weakly almost periodic. That is, corresponding to $\epsilon > 0$, there exists a compact set C in G such that if x is any point of X, every translate of C by an element of G contains an element s such that $g(x, s) \in U_\epsilon(x)$. There exists a compact set B in T such that $T = B + G$. Let z be any point of X. Corresponding to $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in \bar{f}(z, G)$ and $b \in B$, then $f^b(U_\delta(y)) \subset U_\epsilon(f^b(y))$. This follows from the compactness of the sets $\bar{f}(z, G)$ and B. Now g is weakly almost periodic on the minimal set $\bar{f}(z, G)$ and hence there exists a compact set C in G such that if $u \in \bar{f}(z, G)$ then every translate $C + g^*$ of C by an element g^* of G contains an element s such that $g(u, s) \in U_\delta(u)$. We show that the set C satisfies the desired condition with respect to every point x in X. For there exists a point u in $\bar{f}(z, G)$ and an element b in B such that $x = f^b(u)$. Let s be an element in $C + g^*$ such that $f^s(u) \in U_\delta(u)$. But then $f^b(f^s(u)) \in U_\epsilon(f^b(u))$ or $g\,(x, s) \in U_\epsilon(x)$, which is the desired result.

3. **Reducibility and regular almost periodicity.** If we consider all possible relatively dense subgroups of T and the resulting decompositions of X, several possibilities arise. The minimal set X will be said to be *totally minimal* if there exists a point x of X such that for every relatively dense subgroup G of T, $\bar{f}(x, G) = X$. It is clear that if this property holds for one point of X, it holds for all points of X. It is easy to construct examples of totally minimal sets. If X is a circle of circumference unity and α is a positive rational number, the rotations of X about its center through integral multiples of α define a transformation group acting on X with T the discrete group of integers. It is readily verified that in this case X is a minimal set which is totally minimal. More generally, if X is a connected minimal set for which the group T is discrete, and G is a relatively dense subgroup of T, then, since any compact subset of a discrete group is finite, the number of sets in the decomposition of X by G is finite. Since X is connected and these sets are closed and disjoint, there can be only one such set, namely $\bar{f}(x, G)$, and X is totally minimal. However X can be totally disconnected and yet a totally minimal set. Examples of totally minimal sets for which X is the Cantor dis-

continuum and T is the group of integers have been constructed by Hedlund [1, Theorem 6.4, p. 619].

If T is the group of real numbers with its natural topology we say that f defines a *continuous flow* in X. It is possible to display examples of compact manifolds which are totally minimal under continuous flows. A continuous flow in the topological space X is said to be *topologically mixing* if corresponding to any two open subsets A and B of X there exists a real number r such that for $|t|>r, f^t(A)\cap B\neq\emptyset$. It is clear that if X is minimal under a continuous flow which is topologically mixing, then X is totally minimal. Continuous flows in compact three-dimensional manifolds which are minimal and topologically mixing have been constructed by Hedlund ([2, p. 250]. The result is not stated specifically, but it is remarked that if the fundamental region, together with its boundary, lies interior to the unit circle, the horocycle flow has the property that every motion is transitive. Thus the horocycle flow is minimal. It is easy to show that this flow is topologically mixing). These considerations suggest a converse problem, namely as to whether, in the case of a set which is minimal under a continuous flow, total minimality implies that the flow is topologically mixing. The authors have not been able to verify or disprove this conjecture.

In contrast to the case in which X is totally minimal it may happen that corresponding to any $\epsilon>0$ there exists a relatively dense subgroup G of T such that if D is any one of the members of the decomposition of X by G, diam. $D<\epsilon$. Generalizing a definition due to G. T. Whyburn [1, p. 250] we say that in this case the *transformation group T acting on X is regularly almost periodic.* It will also be said in this case that the minimal set X is *completely reducible.*

If the transformation group T is periodic in the sense that there exists a relatively dense subgroup G such that $f(x, G)=x$ for all x in X, then T is regularly almost periodic and the minimal set X is completely reducible. It is not difficult to construct a minimal set X for which T is the group of integers, X is a Cantor discontinuum, and X is completely reducible. In such a case T cannot be periodic in the sense defined.

Again generalizing a definition of Whyburn [1, p. 250] we say that the *point x of X is regularly almost periodic* with respect to T if corresponding to any $\epsilon>0$ there exists a relatively dense subgroup G of T such that diam. $\bar{f}(x, G)<\epsilon$. In this case we say that the minimal set X is *reducible* at x and if every point of X is regularly almost periodic we say that X is *pointwise reducible.* If the minimal set X is completely reducible, it is clearly pointwise reducible. As to whether the converse of this is true in general seems to be difficult to determine, but it is possible to show that pointwise reducibility implies complete reducibility if the group T is discrete. To that end we prove an elementary lemma.

LEMMA 1. *If T is a discrete, abelian, topological group and G_1 and G_2 are relatively dense subgroups of T, then $G_1 \cap G_2$ is a relatively dense subgroup of T, of G_1 and of G_2.*

The proof of this lemma follows readily from the theorem (Kurosch [1, p. 47]) which states that the intersection of a finite number of subgroups of finite index is a group of finite index and the fact that a relatively dense subgroup of T is of finite index.

THEOREM 3. *If X is a compact minimal set which is pointwise reducible and T is discrete, then X is completely reducible.*

For if $\epsilon > 0$ and x is any point of X, there exists a relatively dense subgroup G of T such that diam. $\bar{f}(x, G) < \epsilon/2$. It follows from Theorem 2 that there exists a neighborhood $N(x)$ of x such that if $y \in N(x)$ then diam. $\bar{f}(y, G) < \epsilon$. A finite number of these neighborhoods cover X and let such a set be $N_1, N_2, \cdots, N_m$ with corresponding groups, $G_1, G_2, \cdots, G_m$. According to Lemma 1, the intersection $G^* = G_1 \cap G_2 \cap \cdots \cap G_m$ of these relatively dense subgroups of T is a relatively dense subgroup of T. If z is any point of X, there exists an integer i such that $z \in N_i$ and then we have diam. $\bar{f}(z, G^*) \leqq$ diam. $\bar{f}(z, G_i) < \epsilon$. It follows that X is completely reducible.

If the transformation group T acting on X is regularly almost periodic it is almost periodic and hence the family T is equi-uniformly continuous (cf. Gottschalk [3, Theorem 2, p. 635]).

THEOREM 4. *If X is compact and minimal under the equi-uniformly continuous transformation group T, and if X is reducible at one point, then X is completely reducible.*

Since the group T is equi-uniformly continuous, corresponding to $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $d[f^t(x), f^t(y)] < \epsilon/2$ for all t in T. If $\bar{x}$ is the point at which X is reducible, there exists a relatively dense subgroup G of T such that diam. $\bar{f}(\bar{x}, G) < \delta$. But then if $y = f^t(x)$ is any point in the orbit of x, it follows that $\bar{f}(y, G) = \bar{f}[f^t(\bar{x}), G] = f^t[\bar{f}(\bar{x}, G)]$ and hence diam. $\bar{f}(y, G) \leqq \epsilon/2$. In view of Theorem 2, we see that if x is any point of X, diam. $\bar{f}(x, G) < \epsilon$. The proof of the theorem is complete.

The compact set X, minimal under the transformation group T, will be said to be *0-reducible* if corresponding to $\delta > 0$ there exists a relatively dense subgroup G of T and a point x of X such that diam. $\bar{f}(x, G) < \delta$. If X is a 0-reducible minimal set, each point of X will be said to be *isochronous.*

THEOREM 5. *A necessary and sufficient condition that $y \in X$ be isochronous is that corresponding to $\delta > 0$ there exists a relatively dense subgroup G of T and an element t of T such that* diam. $\bar{f}(y, t+G) < \delta$.

Since $\bar{f}(y, t+G) = \bar{f}[f^t(y), G]$, it is clear that the condition is sufficient. To prove the necessity, suppose that y is isochronous. Then corresponding to $\delta > 0$ there exists a relatively dense subgroup G of T and a point x of X such that diam. $\bar{f}(x, G) < \delta/2$. It follows from Theorem 2 that there exists a neighborhood $N(x)$ of x such that $z \in N(x)$ implies diam. $\bar{f}(z, G) < \delta$. But the orbit of y is everywhere dense in X and thus there exists a t in T such that $f^t(y) \in N(x)$. We now have diam. $\bar{f}(y, t+G) =$ diam. $\bar{f}[f^t(y), G] < \delta$ and the proof of the theorem is complete.

THEOREM 6. *If the compact set X is minimal under T and* 0*-reducible, there exists a point of X at which X is reducible.*

We show first that if y is any point of X and $U_\epsilon(y)$ is any ϵ-neighborhood of y, there exists a point z in $U_\epsilon(y)$ and a relatively dense subgroup G of T such that $\bar{f}(z, G) \subset U_{2\epsilon}(y)$. Since f is weakly almost periodic, corresponding to $\epsilon > 0$, there exists a compact set A in T such that if v is any point of X, each translate of A contains an element t for which $f(v, t) \in U_{\epsilon/2}(y)$. Since X is minimal, there exists an s in T such that $f^s(v) \in U_{\epsilon/2}(v)$. It follows that the set $A-s$ contains an element $t=a-s$, $a \in A$, such that $f^t[f^s(v)] \in U_{\epsilon/2}(f^s(v))$. But then $f^a(v) \in U_{\epsilon/2}[f^s(v)] \subset U_\epsilon(y)$. Thus $f[U_\epsilon(y), -A] = X$. Since the set A is compact, there exists a $\delta > 0$ such that if u and v are any points of X at distance apart less than δ and a is any element of A, then $d[f^a(u), f^a(v)] < \epsilon/2$. Since X is 0-reducible, there exists a point $\bar{x}$ in X and a relatively dense subgroup G of T such that diam. $\bar{f}(\bar{x}, G) < \delta$. Now let a be any element of A such that $f^a(\bar{x}) \in U_\epsilon(y)$. But then diam. $f^a[\bar{f}(\bar{x}, G)] =$ diam. $\bar{f}[f^a(\bar{x}), G] < \epsilon$ and if we let $f^a(\bar{x}) = z$, we have the first of our desired results.

Now let $\epsilon_1 > \epsilon_2 > \cdots$ be a sequence of real numbers with $\lim_{n\to\infty} \epsilon_n = 0$. Let U_1 be a sphere in X with radius ϵ_1. It follows from the first part of the proof that there exists a point x_1 in U_1 and a relatively dense subgroup G_1 of T such that $\bar{f}(x_1, G_1) \subset U_1$. From Theorem 2 we infer the existence of a sphere V_1 with center x_1 such that for every y in $\bar{V}_1$ it is true that $\bar{f}(y, G_1) \subset U_1$. In particular it follows that $\bar{V}_1 \subset U_1$. Now let U_2 be a sphere of radius ϵ_2 in V_1. By a repetition of the argument just given, there exists a point x_2, a sphere V_2 with center x_2, and a relatively dense subgroup G_2 of T such that $\overline{V}_2 \subset U_2 \subset V_1$ and if $z \in \overline{V}_2$ then $\bar{f}(z, G_2) \subset U_2$. Continuing this process we obtain a sequence $V_1 \supset V_2 \supset \cdots$ of spheres such that the radius of V_i is less than ϵ_i, $i = 1, 2, \cdots$, and a sequence of relatively dense subgroups $G_1, G_2, \cdots$ of T such that if x_i is any point of $\overline{V}_i$ then diam. $\bar{f}(x_i, G_i) < \epsilon_i$. Since $\overline{V}_i$ is compact, there must exist a point $x^* \in \bigcap_{i=1}^{\infty} V_i$. Now X is reducible at x^* and the proof of the theorem is complete.

COROLLARY 6.1. *If X is compact and minimal under the equi-uniformly continuous transformation group T and if X is 0-reducible, then X is completely reducible.*

If the group T is discrete it is possible to add several results which are not necessarily true in the general case.

THEOREM 7. *If X is a compact set which is minimal under the discrete transformation group T and G is a relatively dense subgroup of T then*:

(a) *If x is regularly almost periodic under T, x, considered as in $\bar{f}(x, G)$ acted on by G, is regularly almost periodic under G;*

(b) *If x is isochronous under T, x, considered as in $\bar{f}(x, G)$ acted on by G, is isochronous under G.*

Statement (a) of the theorem is a simple application of Lemma 1.

To prove (b) suppose that x is isochronous under T. It follows from Theorem 6 that there exists a point y of X which is regularly almost periodic under T. We infer from (a) that y, considered as in $\bar{f}(y, G)$ acted on by G, is regularly almost periodic under G. But according to Theorem 1, the sets $\bar{f}(x, G)$ and $\bar{f}(y, G)$ are equivalent and there must be a point of $\bar{f}(x, G)$ which is regularly almost periodic under G. This implies that x, considered as in $\bar{f}(x, G)$ acted on by G, is isochronous under G.

4. An isochronous point which is not regularly almost periodic. Let S be the space consisting of the symbols a and b and let I denote the ordered set $\cdots, -1, 0, 1, 2, \cdots$ of all integers. Let X be the set of all mappings from I into S. If x and y are points of X, $x(n)$ denoting the image of n in S under the mapping x, $y(n)$ denoting the image of n in S under the mapping y, we define the distance $d(x, y)$ between x and y as follows:

$$d(x, y) = 0 \qquad \text{if} \quad x(n) = y(n) \text{ for all } n \text{ in } I,$$

$$d(x, y) = 1 \qquad \text{if} \quad x(0) \neq y(0),$$

$$d(x, y) = (k+2)^{-1} \quad \text{if} \quad x(n) = y(n),\ n = 0, \pm 1, \cdots, \pm k,$$
$$\text{and either } x(-k-1) \neq y(-k-1) \quad \text{or} \quad x(k+1) \neq y(k+1).$$

It is easily shown that this metric satisfies the usual metric axioms and that the metric space X thus defined is compact, perfect and totally disconnected (cf. Morse and Hedlund [1, pp. 819–820]). Thus X is a Cantor discontinuum.

Let x be any point of X and let $x(n)$ denote the image of n in S under the mapping x. If to the integer n we let correspond $x(n+1)$ there is defined a mapping of I into S and we denote the point of x thereby defined by y. It is easy to show that the mapping $x \rightarrow y$ is a homeomorphism H of X onto X. This homeomorphism H and its integral powers define a transformation group T acting on X and $T = I$.

We define a sequence of points of X. It will be convenient to term the symbol b the *dual* of a and the symbol a the *dual* of b. If c stands for either a or b, we denote its dual by $\bar{c}$. Now let α_0 be the point of X defined by the mapping

$$\alpha_0(n) = a, \qquad n = 0, \pm 1, \pm 2, \cdots.$$

Proceeding inductively and assuming that $\alpha_0, \alpha_1, \cdots, \alpha_{m-1}$ have been defined, we define α_m by the mapping

$$\alpha_m(n) = \begin{cases} \alpha_{m-1}(n), & n \neq k2^m,\ k = 0, \pm 1, \cdots, \\ \bar{\alpha}_{m-1}(n), & n = k2^m,\ k = 0, \pm 1, \cdots. \end{cases}$$

The point x of X is *periodic* if there exists an ω in I, $\omega \neq 0$, such that $x(n+\omega) = x(n)$ for all n in I. In this case we say that ω is *a period* of x. If x is periodic, there exists a primitive period ω of x in the sense that $\omega \in I$, $\omega > 0$, and any period of x is an integral multiple of ω. We then say that this primitive period is *the period* of x. Clearly x is periodic under T if and only if x is periodic in the sense just defined.

LEMMA 2. *α_m is periodic with the period 2^m.*

We prove this by induction. It follows immediately from its definition that α_0 is periodic with the period 1. Let us suppose that for $0 \leqq i \leqq m-1$, α_i is periodic with the period 2^i. If $n \neq k2^m$, k integral, $\alpha_m(n) = \alpha_{m-1}(n) = \alpha_{m-1}(n+2^{m-1}) = \alpha_{m-1}(n+2^m) = \alpha_m(n+2^m)$. If $n = k2^m$, k integral, we have

$$\alpha_m(n) = \bar{\alpha}_{m-1}(n) = \bar{\alpha}_{m-1}(n + 2^m) = \alpha_m(n + 2^m).$$

Thus 2^m is a period of α_m. If 2^m is not the period of α_m and ω is the period of α_m, then 2^m must be a multiple of ω and ω must be a power of 2. It follows in particular that 2^{m-1} is a period of α_m. But

$$\alpha_m(0) = \bar{\alpha}_{m-1}(0) = \bar{\alpha}_{m-1}(2^{m-1}) = \bar{\alpha}_m(2^{m-1}) = \bar{\alpha}_m(0).$$

From this contradiction we infer that 2^{m-1} is not a period of α_m and 2^m must be the period of α_m. The proof of the lemma is complete.

Remark 1. If k and m are integers such that $k>m$, then $\alpha_k(n) = \alpha_m(n)$ provided $n \neq p2^{m+1}$, p integral.

This follows immediately from the definition of α_m.

Remark 2. The sequence of points $\alpha_0, \alpha_2, \alpha_4 \cdots$ converges in X.

For if k and m are integers such that $k>m$,

$$\alpha_{2k}(n) = \alpha_{2m}(n), \qquad n = 0, \pm 1, \cdots, \pm(2^{2m+1} - 1).$$

If we define β by the mapping

$$\beta(n) = \alpha_{2m}(n), \qquad |n| < 2^{2m+1} - 1$$

then $\beta(n)$ is uniquely defined and it is clear that $\alpha_{2m} \to \beta$.

Remark 3. If x is periodic with the period ω, there cannot exist integers k and ω_1 such that $0<\omega_1<\omega$ and

$$x(n + \omega_1) = x(n), \qquad n = k + 1, k + 2, \cdots, k + \omega. \tag{4}$$

For suppose (4) were true. If p is integral, there exists an integer q such that $p+q\omega = k+i$ with $1 \leqq i \leqq \omega$. But now we have

$$x(p) = x(p + q\omega) = x(k + i) = x(k + i + \omega_1) = x(p + \omega_1).$$

Thus x has ω_1 as period, contrary to the hypothesis that the period of x is ω.

LEMMA 3. *β is not periodic.*

For suppose that β has the period ω. Let the positive integer m be so chosen that $2^{2m} > \omega$. According to the definition of β,

$$\beta(n) = \alpha_{2m}(n), \qquad n = 0, \pm 1, \cdots, \pm(2^{2m+1} - 1)$$

and if β has period ω we have

$$\begin{gathered} \alpha_{2m}(k + \omega) = \alpha_{2m}(k), \\ k = -2^{2m+1} + 1, -2^{2m+1} + 2, \cdots, -2^{2m+1} + 2^{2m}. \end{gathered} \tag{5}$$

Since α_{2m} has the period 2^{2m}, it follows from Remark 3 that (5) is impossible. We infer that β cannot be periodic.

Remark 4. $\beta(n) = \alpha_{2m}(n)$, $n \neq k2^{2m+1}$, $k = \pm 1, \pm 2, \cdots$.

LEMMA 4. *Given $\epsilon > 0$ there exist integers l and $k > 0$ such that*

$$d(\beta, H^{l+pk}(\beta)) < \epsilon, \qquad p \in I.$$

Let m be a positive integer such that $2^m < 1/\epsilon$, let $l = 2^{2m}$ and let $k = 2^{2m+1}$. Then if $p \in I$ and $|n| < 2^m$, we have

$$\begin{aligned} \beta(l + pk + n) &= \beta(2^{2m} + p2^{2m+1} + n) = \alpha_{2m}(2^{2m} + p2^{2m+1} + n) \\ &= \alpha_{2m}(n) = \beta(n). \end{aligned}$$

This implies

$$d(\beta, H^{l+pk}(\beta)) \leqq 2^{-m} < \epsilon$$

and the lemma is proved.

Since the set pk, $p = 0, \pm 1, \cdots$ is relatively dense in I, it follows from Lemma 4 that β is an almost periodic point. Thus if $\Gamma(\beta)$ denotes the orbit closure of β in X, $\Gamma(\beta)$ is minimal under the transformation group T. From Lemma 4 and Theorem 5 we infer that β is isochronous and thus $\Gamma(\beta)$ is 0-reducible. From Theorem 6 there exists a point γ of $\Gamma(\beta)$ at which $\Gamma(\beta)$ is reducible and γ is regularly almost periodic with respect to T.

But $\Gamma(\beta)$ is not completely reducible. For if this were the case the transformation group T acting on X would be regularly almost periodic and hence equi-uniformly continuous. Since β is almost periodic but not periodic, there exists a sequence of points $\beta_1, \beta_2, \cdots$ of $\Gamma(\beta)$ such that $\beta_i \to \beta$ and $\beta_i \neq \beta$, $i = 1, 2, \cdots$. Since $\beta_i \neq \beta$, there exists an integer n such that $\beta_i(n) \neq \beta(n)$. This implies that $d[H^n(\beta_i), H^n(\beta)] = 1$. Thus if $\epsilon = 1/2$, there exists no $\delta > 0$ such that $d(\beta_i, \beta) < \delta$ implies $d[H^p(\beta_i), H^p(\beta)] < \epsilon$ for all p in I. It follows that the transformation group T acting on $\Gamma(\beta)$ is not equi-uniformly continuous and $\Gamma(\beta)$ is not completely reducible.

Since T is discrete, it follows from Theorem 3 that $\Gamma(\beta)$ is not pointwise reducible. Since $\Gamma(\beta)$ is a minimal set, the orbit closure $\Gamma(\gamma)$ of γ is identical with $\Gamma(\beta)$. Thus γ is a regularly almost periodic point with the property that not all points in the orbit closure of γ are regularly almost periodic.

Bibliography

W. H. Gottschalk

1. *Almost periodic points with respect to transformation semi-groups*, Ann. of Math. vol. 47 (1946) p. 762.
2. *Almost periodicity, equi-continuity and total boundedness*, Bull. Amer. Math. Soc. vol. 52 (1946) p. 633.

G. A. Hedlund

1. *Sturmian minimal sets*, Amer. J. Math. vol. 66 (1944) p. 605.
2. *The dynamics of geodesic flows*, Bull. Amer. Math. Soc. vol. 45 (1939) p. 241.

A. Kurosch
1. *Theory of groups* (in Russian), Moscow, 1944.
Marston Morse and G. A. Hedlund
1. *Symbolic dynamics*, Amer. J. Math. vol. 60 (1938) p. 815.
G. T. Whyburn
1. *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, New York, 1942.

The University of Puerto Rico and
The University of Virginia

A ZERO-DIMENSIONAL TOPOLOGICAL GROUP WITH A ONE-DIMENSIONAL FACTOR GROUP

SAMUEL KAPLAN

As can be easily shown, if a locally compact topological group is zero-dimensional, all of its factor groups are zero-dimensional. In this note we give an example of a non locally compact zero-dimensional group with a factor group which is topologically isomorphic to the real numbers, hence one-dimensional.[1]

1. **Preliminaries.** Let $\{\lambda\}$ be a set of indices of cardinality c, and for each λ, let R_λ be a topological isomorph of the additive group of rational numbers. We form the weak product R of the R_λ: an element r of R is a collection $r=\{r_\lambda\}$, $r_\lambda \in R_\lambda$, such that for only a finite number of the λ's is $r_\lambda \neq 0_\lambda$. Under the definitions $r+r'=\{r_\lambda+r'_\lambda\}$, $0=\{0_\lambda\}$, R forms a group.

Now for each $r\in R$, we define $\|r\| = \sum_\lambda |r_\lambda|$. Since all but a finite number of the $r_\lambda = 0_\lambda$, this sum exists. Clearly $\|r+r'\| \leqq \|r\|+\|r'\|$, and $\|-r\| = \|r\|$, hence, as can be easily shown, $\|r\|$ defines a metric in R under the definition: the distance from r to r' is $\|r-r'\|$.

Lemma 1. *Let $\{d_\lambda\}$ be a set of positive real numbers bounded away from zero, that is, there exists $d>0$ such thal $d_\lambda \geqq d$ for all λ. Then*

$$U = \left\{ r \,\middle|\, \sum_\lambda \left| \frac{r_\lambda}{d_\lambda} \right| < 1 \right\}$$

Received by the editors November 10, 1947.

[1] Cf. Bourbaki, *Topologie generale*, chap. III, p. 21, exercise 12, for an example of a totally disconnected group with a factor group topologically isomorphic to the reals. This example was pointed out to me by I. Kaplansky.