A NOTE ON THE SINGULAR MANIFOLDS
OF A DIFFERENCE POLYNOMIAL

RICHARD M. COHN

1. Introduction. In a previous paper[1] we defined essential singular manifolds of a difference polynomial in one unknown, and gave an example of such a manifold. By an obvious extension of this definition we may say that if A is an algebraically irreducible difference polynomial in unknowns $y_1, \cdots, y_n$, then an essential irreducible manifold of A which annuls a polynomial of lower effective[2] order than A in y_k, $1 \leq k \leq n$, or free of y_k is an *essential singular manifold of A relative to y_k.* The remaining essential irreducible manifolds of A we shall call, as in the case of a polynomial in one unknown, *ordinary manifolds relative to y_k,* and the totality of solutions in these manifolds the general solution *of A relative to y_k,*

The analogous situation in the theory of algebraic differential equations[3] suggests that the essential singular manifolds of a difference polynomial relative to one unknown are also essential singular manifolds relative to any other unknown. It is the purpose of this paper to show that this is actually the case. It will follow that we may drop the term "relative" from the concepts we have just defined. *The essential irreducible manifolds of an algebraically irreducible difference polynomial may be divided into two classes, singular manifolds and ordinary manifolds. The singular manifolds are, in the sense defined above, singular relative to each unknown present in the difference polynomial. The ordinary manifolds are ordinary relative to each unknown, and the totality of solutions they contain may be called the general solution of the difference polynomial.*

We make use, as in the theory of algebraic differential equations, of the *separants*[4] of a difference polynomial. Let A be a difference

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[1] *Manifolds of difference polynomials*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 133–172, referred to below as M.D.P.; §21.

[2] The effective order of a difference polynomial in y_k is defined in M.D.P. as the difference between the orders of the highest and lowest transforms of y_k appearing effectively in the polynomial.

[3] J. F. Ritt, *Differential equations from the algebraic standpoint*, Amer. Math. Soc. Colloquium Publications, vol. 14, 1932, p. 24.

[4] The perhaps unexpected fact that the separant plays a rôle in the theory of difference equations was observed by Poisson, *Mémoire sur les solutions particulières des équations differentielles et des équations aux différences*, J. École Polytech. vol. 6 (1806) pp. 60–125. Poisson's "particular solutions" do not necessarily lie in essential singular manifolds.

polynomial in $y_1, \cdots, y_n$. The y_k-separant of A is defined as the formal partial derivative $\partial A/\partial y_{kr_k}$, where r_k is the order of A in y_k. It will be easy to derive the result stated above from the following theorem.

THEOREM. *Let A be a difference polynomial in unknowns $y_1, \cdots, y_n$. An essential irreducible manifold of A which is singular relative to y_k annuls the y_k-separant of A.*

We proceed to prove this theorem.

2. A lemma on algebraic polynomials. We shall need solutions of difference polynomials in the form of formal power series expansions. We shall first state and prove a lemma concerning the existence of such solutions for algebraic[5] polynomials.

LEMMA. *Let P be an algebraic polynomial in unknowns $x_1, \cdots, x_n$, z with coefficients in an algebraic field K. Let P vanish, and its formal partial derivative $\partial P/\partial z$ not vanish, when $x_1, \cdots, x_n$, z are set equal to zero. Then there exists a formal series in positive integral powers of $x_1, \cdots, x_n$ with coefficients in K which annuls P formally when it is substituted for z.*

We write down the equations which result from the formal differentiation of $P = 0$, when z is considered as a function of $x_1, \cdots, x_n$.

$$\frac{\partial P}{\partial z}\frac{\partial z}{\partial x_1}+\frac{\partial P}{\partial x_1}=0, \qquad \frac{\partial P}{\partial z}\frac{\partial z}{\partial x_2}+\frac{\partial P}{\partial x_2}=0,$$

(1)
$$\cdots\cdots\cdots\cdots,$$

$$\frac{\partial P}{\partial z}\frac{\partial z}{\partial x_n}+\frac{\partial P}{\partial x_n}=0.$$

When we substitute zero for $x_1, \cdots, x_n$, z in these equations they determine, since $\partial P/\partial z$ does not vanish, values for $\partial z/\partial x_i$, $i=1, \cdots, n$. These values lie in K.

We differentiate formally the equations (1) obtaining a system of equations of the form

$$\frac{\partial P}{\partial z}\frac{\partial^2 z}{\partial x_j\partial x_i}+\frac{\partial^2 P}{\partial x_j\partial z}\frac{\partial z}{\partial x_i}+\frac{\partial^2 P}{\partial x_j\partial x_i}$$

(2)
$$+\frac{\partial z}{\partial x_j}\left(\frac{\partial^2 P}{\partial z^2}\frac{\partial z}{\partial x_i}+\frac{\partial^2 P}{\partial z\partial x_i}\right)=0, \qquad 1 \leqq i, j \leqq n; i \leqq j.$$

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[5] That is, polynomials in the usual sense, not difference polynomials.

Letting $x_1, \cdots, x_n, z$ be zero in these equations, and giving the partial derivatives $\partial z/\partial x_i$ the values obtained in (1), we obtain unique values lying in K for each $\partial^2 z/\partial x_i^2$ and $\partial^2 z/\partial x_i \partial x_j$, $i < j$.

We obtain further equations by differentiating the equations of (2). We differentiate each equation of (2) once with respect to each x_i and retain one equation from each set of equations which differ only in the order of taking the derivatives. These equations we call (3). We treat the system (3) as (2) was treated forming (4). We continue this process and consider the system λ consisting of the systems (1), (2), (3), $\cdots$.

From the equations (3) we obtain values of the derivatives of z of third order by letting $x_1, \cdots, x_n, z$ be zero and assigning to derivatives of first and second order the values already computed. In this way we obtain the derivatives of z of all orders successively from the equations of λ. Evidently these will all be elements of K.

Using the computed values of the partial derivatives of z we form the Taylor's series z' for z in powers of the x_i. We shall show that this series has the properties stated in the lemma.

It is only necessary to prove that z' annuls P formally when substituted for z. This will follow if we can show that, for every m, when z is replaced in P by the polynomial z_m consisting of all terms of z' of degree less than m, the resulting polynomial $P(z_m)$ has no terms of degree less than m in the x_i. But this in turn is equivalent to showing that, if we take the formal partial derivatives of $P(z_m)$ with respect to the x_i, all derivatives of order less than m, as well as $P(z_m)$ itself, vanish when the x_i are set equal to zero. Now the equations expressing this condition are obtained from the equation $P=0$ and those equations of the system λ which involve no derivatives of order exceeding $m-1$ by replacing z by z_m and setting $x_1, \cdots, x_n$ equal to zero. But these equations must be valid, for the partial derivatives of z_m of order less than m are identical, when the x_i are zero, with the partial derivatives of z'; and the Taylor's series z' was so constructed that its derivatives satisfy the equations λ and $P=0$ under the stated condition. Thus the lemma is proved.

3. **Proof of the theorem.** We turn now to the proof of the theorem on separants. We consider an algebraically irreducible difference polynomial A with coefficients in a difference field $\mathcal{J}$. The unknowns in A we shall now denote, for convenience, by the letters $u_1, \cdots, u_q; y$. We consider an irreducible manifold $\mathfrak{M}$ annulling A, but not $\partial A/\partial y_r$, where r is the order of A in y. We shall prove that $\mathfrak{M}$ is contained in or coincides with one of the ordinary manifolds of A relative to y.

We may assume that the order and effective order of A in y are equal. For, if not, we could replace y by a new variable $\bar{y} = y_k$, where k is the difference between the order and effective order. A would thus be transformed into a polynomial of equal order and effective order in $\bar{y}$; and the other conditions of the problem remain unaltered.

We denote by Λ the reflexive prime ideal consisting of all polynomials which vanish for all solutions in $\mathfrak{M}$. Let $y = \alpha$, $u_i = \gamma_i$, be a general point of Λ. Upon making the substitution $y = z + \alpha$, $u_i = v_i + \gamma_i$, A becomes a difference polynomial $\bar{A}$ in the v_i and z, which vanishes when z and the v_i are set equal to zero; while $\partial A / \partial y_r$ is carried into the polynomial $\partial \bar{A} / \partial z_r$ which does not vanish with z and the v_i. It follows from the lemma that $\bar{A}$ can be annulled formally by substituting for z_r a series z_r' in positive integral powers of z, z_1, $\cdots$, z_{r-1}, and certain v_{ij}. The coefficients in z_r' lie in the field G obtained by adjoining α and the γ_i to $\mathcal{J}$.

We now construct a formal powers series from z_r' by the following procedure. First we replace each coefficient by its transform in G. Next we replace z_i by z_{i+1}, $i = 0, 1, \cdots, r-1$, throughout the series. Finally we replace each v_{ij} by $v_{i,j+1}$. The resulting series we shall call z_{r+1}''. We substitute for z_r in the series z_{r+1}'' the series z_r'. We obtain thus a series z_{r+1}' in positive integral powers of z, z_1, $\cdots$, z_{r-1} and certain v_{ij}, with coefficients in G.

Let the series z_{r+2}'' result from z_{r+1}' by replacing each coefficient by its transform, each v_{ij} by $v_{i,j+1}$, and then replacing each z_i, $0 \leq i \leq r-1$, by z_{i+1}. From z_{r+2}'' we obtain z_{r+2}' by replacing z_r in the expansion by z_r'. We continue in this way to construct series z_i', $i = r$, $r+1$, $r+2$, $\cdots$, in powers of z, z_1, $\cdots$, z_{r-1} and the v_{ij}, and series z_i'', $i = r+1$, $r+2$, $r+3$, $\cdots$ in powers of z_1, z_2, $\cdots$, z_r and the v_{ij}. The coefficients of each z_i' and z_i'' lie in G.

Let $y_i' = z_i + \alpha_i$, $0 \leq i < r$, and $y_i' = z_i' + \alpha_i$, $i \geq r$. Let $y_i'' = z_i + \alpha_i$, $1 \leq i < r+1$, and $y_i'' = z_i'' + \alpha_i$, $i \geq r+1$. Finally let $u_{ij}' = u_{ij}'' = v_{ij} + \gamma_{ij}$, $1 \leq i \leq q$, $j \geq 0$.

We consider any polynomial C in the unknowns $u_1, \cdots, u_q$; y with coefficients in the field $\mathcal{J}$. The result of substituting y_i' for y_i and u_{ij}' for u_{ij} in C and its transform C_1 is the pair of formal power series C' and C_1' with coefficients in G. Let C_1'' be the expansion which results from C_1 upon making the substitution of y_i'' for y_i, $i \geq 1$, and u_{ij}'' for u_{ij}. Evidently C_1'' can be constructed from C' by replacing z_i by z_{i+1}, $0 \leq i < r$, v_{ij} by $v_{i,j+1}$, and each coefficient of the expansion by its transform. C_1' may now be obtained from C_1'' by replacing z_r by z_r'. We see that, if C' vanishes identically, so does C_1'', and therefore C_1'. It follows that the set of all polynomials

which vanish when the y_i and the u_{ij} are replaced by the y_i' and u_{ij}' forms a difference ideal Ψ. Evidently Ψ is prime and contains A.

Let Σ be the set of all polynomials whose transforms of any order are in Ψ. Evidently Σ is a prime reflexive difference ideal. We shall show that no polynomial of order less than r in y, or free of y, holds Σ. First we consider a polynomial D of order not exceeding $r-1$ in y and show that it is not in Ψ. The substitutions $y_i = y_i'$, $u_{ij} = u_{ij}'$ become, in this special case, a linear transformation with an inverse. Since D is not identically zero, the polynomial into which it is carried is not zero. Thus D is not in Ψ. We now assume that all polynomials of effective order not exceeding $r-1$ and order not exceeding $r-1+k$ are not in Ψ. We shall show that a polynomial D of order $r+k$ and effective order not exceeding $r-1$ in y is not in Ψ. We have shown elsewhere[6] that the resultant R with respect to y_{r+k} of D and A_k is a nonzero polynomial of order at most $r-1+k$ and effective order at most $r-1$. Our assumption shows that R is not annulled by the substitutions in question, so that D cannot be annulled by them. We can now show by induction that no polynomial of effective order at most $r-1$ in y is in Ψ, so that no polynomial of order not exceeding $r-1$ in y is in Σ. It follows that Σ is one of the reflexive prime ideals whose solutions constitute the ordinary manifolds of A.

Some transform of any polynomial of Σ must be annulled by the y_i' and u_{ij}'. Now on substituting these quantities for the y_i and u_{ij} in the transform C_k of a polynomial C a term of zero degree is obtained which is equal to the result of substituting α for y and γ_i for u_i in C_k. Since this term must vanish for every polynomial C of Σ and a suitable k depending on C, C_k and therefore C is annulled by the general point of Λ. Then Σ holds Λ. This proves the theorem.

4. **Proof of the principal result.** We now return to the proof of the possibility of the division of the essential irreducible manifolds of a difference polynomial into singular and ordinary manifolds. We consider an algebraically irreducible difference polynomial A in unknowns $y_1, \cdots, y_n$ with coefficients in a difference field $\mathcal{F}$. Let $\mathfrak{M}$ be an essential irreducible manifold of A, and let Λ be the corresponding reflexive prime ideal. We wish to show that $\mathfrak{M}$ is either an essential singular manifold of A relative to each y_i or an ordinary manifold relative to each y_i. For this purpose it will suffice to assume that $\mathfrak{M}$ is an essential singular manifold relative to y_1, and prove that it is singular relative to y_2.

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[6] M. D. P., §14. The presence of u_{ij}, which are not included in the discussion of M.D.P., does not affect the argument.

If we make this assumption it follows that $\mathfrak{M}$ annuls $\partial A/\partial y_{1r}$, where r is the order of A in y_1. Let s be the order of A in y_2. We form the resultant R of A and $\partial A/\partial y_{1r}$, considered as algebraic polynomials in y_{2s}. Since A is irreducible, and cannot be a factor of $\partial A/\partial y_{1r}$, R is a nonzero polynomial, free of y_{2s}, which is annulled by $\mathfrak{M}$. Since R is of lower efective order than A in y_2, $\mathfrak{M}$ must be an essential singular manifold of A relative to y_2. The proof is now complete.

RUTGERS UNIVERSITY

DISTINCT REPRESENTATIVES OF SUBSETS

MARSHALL HALL, JR.

1. Introduction. Let W be a set of elements a_i $\cdot$ $W = \{a_1, \cdots\}$ and let $U\{S_1, \cdots, S_j, \cdots\}$ be an indexed system of subsets of W. We wish to choose distinct representatives of the subsets. If $a_j = R(S_j)$ designates the representative of the subset S_j, then we require $R(S_j) \in S_j$ for all j and $R(S_j) \neq R(S_k)$ if $j \neq k$. It is to be emphasized that subsets are distinguished only by their indices and distinct subsets may contain the same elements. An obviously necessary condition for the existence of distinct representatives is:

Condition C: *Every k distinct subsets contain between them at least k distinct elements, for every finite k.* P. Hall[1] has shown that if the number of subsets is finite, condition C is also sufficient for the existence of a system of distinct representatives, or SDR as we shall abbreviate. This condition is no longer sufficient if the number of subsets is infinite. As a counter example consider $U(S_0, S_1, \cdots\}$ where $S_0 = \{a_1, a_2, \cdots\}$, $S_i = \{a_i\}$, $i = 1, 2, \cdots$. Here condition C is easily shown to hold for the subsets, but clearly no representative may be selected for S_0 which is not also a representative of some S_i.

In this paper it is shown that condition C is sufficient if every subset S_j is finite, and also an estimate on the number of systems of distinct representatives is given. This latter result is applied to Latin squares.

THEOREM 1. *Given an indexed system $U\{S_1, \cdots, S_j, \cdots\}$ of finite subsets of a set $W\{a_1, \cdots, a_i, \cdots\}$. If the subsets satisfy condi-*

[1] P. Hall, *On representatives of subsets*, J. London Math. Soc. vol. 10 (1935) pp. 26–30.