SOME ANALOGS OF THE GENERALIZED PRINCIPAL AXIS TRANSFORMATION

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It is known that two normal matrices can be diagonalized by the same unitary transformation if and only if they commute; this theorem is ordinarily stated for hermitian matrices. Some generalizations of this theorem are known. According to a theorem due to Eckert and Young,[1] if A and B are two $r\times s$ matrices, there are two unitary matrices U and V such that $UAV=D_1$ and $UBV=D_2$, D_1 and D_2 diagonal matrices with real elements, if and only if AB^{ct} and $B^{ct}A$ are hermitian. It is also known that a set of normal matrices $\{A_i\}$ is reducible to diagonal matrices under the same unitary similarity transformation, UA_iU^{ct}, if and only if $A_iA_j=A_jA_i$ for all i and j. (More generally, it is true that a set of matrices $\{A_i\}$ with elements in the complex field and simple elementary divisors is reducible to diagonal matrices under the same similarity transformation if and only if $A_iA_j=A_jA_i$ for all i and j.) The following will be shown to hold:

THEOREM. *If $\{A_i\}$ is an arbitrary set of nonzero $r\times s$ matrices, there are unitary matrices U and V of orders $r\times r$ and $s\times s$, respectively, such that $UA_iV=D_i$, D_i diagonal and real, if and only if $A_iA_j^{ct}=A_jA_i^{ct}$ and $A_j^{ct}A_i=A_i^{ct}A_j$ for all i and j.*

If two unitary matrices U and V exist such that $UA_iV=D_i$, D_i real for all i, then $D_iD_j^{ct}=D_iD_j^{t}=D_jD_i^{t}=D_jD_i^{ct}$ where the D_i are $r\times s$ diagonal matrices (that is, the only nonzero elements appear in the d_{ii} position). Therefore, $A_iA_j^{ct}=A_jA_i^{ct}$.

Conversely, let the relations $A_j^{ct}A_i=A_i^{ct}A_j$ and $A_iA_j^{ct}=A_jA_i^{ct}$ hold for all i, j. The proof is by induction.

(1) The theorem is true for a set of matrices of dimension $1\times s$, $A_i=[a_i', a_i'', \cdots, a_i^{(s)}]$. For there exist unitary matrices U and V such that[1] $UA_1V=[d_1', 0, \cdots, 0]$ for d_1' real and greater than 0 since $A_1\neq 0$. For if $UA_iV=[d_i', d_i'', \cdots, d_i^{(s)}]$, it follows from $A_i^{ct}A_1=A_1^{ct}A_i$ that $d_i''=d_i'''=\cdots=d_i^{(s)}=0$ and since $d_1'\cdot\bar{d}_i'=\bar{d}_1'\cdot d_i'$ and d_1' is real, $\bar{d}_i'=d_i'$. In the same way by means of the second of

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[1] Bull. Amer. Math. Soc. vol. 45 (1939) pp. 118–121. See also J. Williamson, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 920–922.

the given conditions, the theorem is true for a set of matrices of dimension $r\times 1$.

(2) Assume the theorem to be true for a set of matrices of dimension $k\times l$ for $k\leqq r$, $l\leqq s-1$ and for $k\leqq r-1$, $l\leqq s$. The theorem will be shown to hold for the dimension $r\times s$ and the induction will then be complete. Let $\{A_i\}$ be a set of matrices of dimension $r\times s$ for which the given conditions hold. Let U and V be such that

$$UA_1V=\begin{bmatrix} D & 0_2 \\ 0_3 & 0_4 \end{bmatrix}$$

where U and V are unitary, D a nonsingular diagonal matrix with real positive diagonal elements, and the submatrices 0_2, 0_3, and 0_4 are null matrices or non-existent. If

$$UA_iV=\begin{bmatrix} G_i & K_i \\ L_i & H_i \end{bmatrix}$$

it follows from $A_1A_i^{ct}=A_iA_1^{ct}$ that $L_i=0_3$, and from $A_i^{ct}A_1=A_1^{ct}A_i$ that $K_i=0_2$. Also, $DG_i^{ct}=G_iD$ and $G_i^{ct}D=DG_i$ from both given conditions. Therefore, $D^2G_i^{ct}=DG_iD=G_i^{ct}D^2$ so $D^2G_i=G_iD^2$. Since D consists of positive real numbers and since G_i commutes with D^2, $DG_i=G_iD$. Then $DG_i^{ct}=G_iD=DG_i$ and since D is nonsingular, $G_i^{ct}=G_i$ for all i. Then from the given relation $A_iA_j^{ct}=A_jA_i^{ct}$, it follows that $G_iG_j^{ct}=G_jG_i^{ct}$ or $G_iG_j=G_jG_i$; therefore the set of hermitian matrices $\{D, G_i\}$ are all commutative in pairs and, by the generalized principal axis theorem, there exists a unitary matrix U_1 which diagonalizes all of them. Let $U_2=U_1\dotplus I$ be a unitary matrix of the same dimension as U and $U_3=U_1^{ct}\dotplus I$ of the same dimension as V. Then,

$$U_2UA_1VU_3=\begin{bmatrix} D & 0_2 \\ 0_3 & 0_4 \end{bmatrix},\quad U_2UA_iVU_3=\begin{bmatrix} D_i & 0_2 \\ 0_3 & H_i \end{bmatrix}$$

for all i where the H_i are either non-existent or of dimension $k\times l$ where $k<r$ and $l<s$. The theorem follows from the induction hypothesis.

It is to be noted that if the set $\{A_i\}$ are all $n\times n$ hermitian matrices for which $A_iA_j^{ct}=A_jA_i^{ct}$ or $A_iA_j=A_jA_i$ holds, the principal axis transformation for hermitian matrices is obtained and $V=U^{ct}$.

According to another result due to Eckert and Young,[1] if A and B are $r\times s$ matrices over the complex field, a necessary and sufficient condition that there exist two unitary matrices U and V such that $UAV=D_1$ and $UBV=D_2$, D_1 and D_2 diagonal, is that AB^{ct} and $B^{ct}A$ be normal. Since this is a generalization of the earlier result, it would seem reasonable to hope for an extension to a set of matrices $\{A_i\}$. A simple example shows that this is not the case, however, and the following theorem holds:

THEOREM. *A necessary and sufficient condition that a set of $n\times n$ matrices $\{A_i\}$ be brought into diagonal forms by the same unitary U, V equivalence transformation, $UA_iV=D_i$, is that the products $A_iA_j^{ct}$ and $A_j^{ct}A_i$ be normal for all i, j and that $A_k(A_j^{ct}A_i)=(A_iA_j^{ct})A_k$ for all i, j and k.*

If $UA_iV=D_i$ for all i, then the given conditions can be easily verified.

Conversely, let $\{A_i\}$ be a set of matrices for which $A_iA_j^{ct}$ and $A_j^{ct}A_i$ are normal and where $A_k(A_j^{ct}A_i)=(A_iA_j^{ct})A_k$. The proof is by induction on the order n. The theorem is trivially true if $n=1$. Assume it to be true for order $k\leqq n-1$. Now consider a system of order n. There are two possibilities: for all i, j, either $A_iA_j^{ct}$ is a scaler matrix or there is at least one pair i, j such that $A_iA_j^{ct}$ is not a scaler.

(1) If for all i, j, $A_iA_j^{ct}$ is a scaler, $A_iA_j^{ct}=k_{ij}I$ and since $A_iA_j^{ct}$ is similar to $A_j^{ct}A_i$, it is true that:

$$\text{(a)} \qquad A_iA_j^{ct} = k_{ij}I = A_j^{ct}A_i \qquad \text{for all } i, j.$$

There are two possibilities: (a) Either all $A_i=k_iU_i$ where the k_i are real positive scalers and U_i are unitary; then all A_i are normal and from the above, $A_iA_j=A_jA_i$ since $A_j=f(A_j^{ct})$. In this case the principal axis transformation theorem applies for normal matrices so $V=U^{ct}$ and the theorem is true. (b) There is at least one A_i, say A_1, not of the above form. There exist two unitary U, V such that $UA_1V=D_1$ is diagonal with real non-negative elements. Also, D_1 is not scaler for then $A_1=U^{ct}D_1V^{ct}=D_1U^{ct}V^{ct}$; but this contradicts the assumption. Let $UA_jV=A_j'$. Then,

$$\text{(a):} \qquad \begin{aligned} UA_1VV^{ct}A_j^{ct}U^{ct} &= Uk_{1j}U^{ct} = k_{1j}I = D_1A_j'^{ct}; \\ V^{ct}A_j^{ct}U^{ct}UA_1V &= V^{ct}k_{1j}V = k_{1j}I = A_j'^{ct}D_1. \end{aligned}$$

Therefore,

$$D_1A_j'^{ct} = A_j'^{ct}D_1$$

and

$$A_j'D_1 = D_1A_j'.$$

Since D_1 is not scaler, the A_j' are direct sums of matrices of order

$k \leqq n-1$. But for these matrices the given conditions hold and the theorem is true.

(2) If for some i, j the products $A_i A_j^{ct}$ (and consequently $A_j^{ct} A_i$) are not scaler, there exist for this A_i and A_j unitary matrices U_{ij} and V_{ij} such that

$$U_{ij} A_i V_{ij} = D_i, \qquad U_{ij} A_j V_{ij} = D_j.$$

For all k, $A_k(A_j^{ct} A_i) = (A_i A_j^{ct}) A_k$.

Apply the U_{ij}, V_{ij} and obtain

$$U_{ij} A_k (A_j^{ct} A_i) V_{ij} = U_{ij} (A_i A_j^{ct}) A_k V_{ij}$$

so

$$U_{ij} A_k V_{ij} V_{ij}^{ct} A_j^{ct} U_{ij}^{ct} U_{ij} A_i V_{ij} = U_{ij} A_i V_{ij} V_{ij}^{ct} A_j^{ct} U_{ij}^{ct} U_{ij} A_k V_{ij},$$

so

$$(U_{ij} A_k V_{ij})(D_j^{ct} D_i) = (D_i D_j^{ct})(U_{ij} A_k V_{ij}).$$

Therefore, the matrix $U_{ij} A_k V_{ij}$ commutes with the nonscaler diagonal matrix $D_j^{ct} D_i$. Since the unitary transformation may be chosen so that like elements of $D_j^{ct} D_i$ appear together in order, $U_{ij} A_k V_{ij}$ is a direct sum of matrices of order $m < n$ for all k. Since the submatrices satisfy these conditions, the theorem is true by induction.

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