CONCERNING AUTOMORPHISMS OF NON-ASSOCIATIVE ALGEBRAS

R. D. SCHAFER

In their studies of non-associative algebras A. A. Albert and N. Jacobson have made much use of the relationships which exist between an arbitrary non-associative algebra $\mathfrak{A}$ and its associative transformation algebra $T(\mathfrak{A})$. In this paper we are interested in the automorphism group $\mathfrak{G}$ of $\mathfrak{A}$, and we sharpen the results of Jacobson [**3**, §4][1] and Albert [**2**, §9] in the sense that we prove $\mathfrak{G}$ isomorphic to a *well-defined* subgroup of the automorphism group of each of three associative algebras (§§2, 3).

Incidental to our proofs is the reconstruction (in the sense of equivalence) of an arbitrary non-associative algebra $\mathfrak{A}$ with unity element 1 from $T(\mathfrak{A})$ and from either of the enveloping algebras $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$ of respectively the right or left multiplications of $\mathfrak{A}$. This paper has been expanded in accordance with suggestions of the referee to include a more detailed study of the right ideals used in this reconstruction process (§5).

1. **Preliminaries.** Our notations are chiefly those of Albert as given in [**1**]. We regard a non-associative algebra $\mathfrak{A}$ of order n over a field $\mathfrak{F}$ as consisting of a linear space $\mathfrak{L}$ of order n over $\mathfrak{F}$, a linear space $R(\mathfrak{A})$ of linear transformations R_x on $\mathfrak{L}$ of order $m \leqq n$ over $\mathfrak{F}$, and a linear mapping of $\mathfrak{L}$ on $R(\mathfrak{A})$,

$$x \rightarrow R_x. \tag{1}$$

The elements R_x of $R(\mathfrak{A})$ are called *right multiplications*, and $R(\mathfrak{A})$ the *right multiplication space* of $\mathfrak{A}$. Multiplication in $\mathfrak{A}$ is defined by

$$a \cdot x = aR_x. \tag{2}$$

The linearity of the right multiplications and of (1) insures distributivity in $\mathfrak{A}$ as well as the usual laws of scalar multiplication. We shall use the fact that, in case $\mathfrak{A}$ contains no absolute right divisor of zero (an element x such that $a \cdot x = 0$ for all a in $\mathfrak{A}$), the mapping (1) is nonsingular and the order of $R(\mathfrak{A})$ over $\mathfrak{F}$ is n.

The linear transformations L_x defined by

$$a \rightarrow x \cdot a = aL_x \tag{3}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

are called *left multiplications* of $\mathfrak{A}$ and form the *left multiplication space* $L(\mathfrak{A})$ of $\mathfrak{A}$. The algebra $\mathfrak{A}$ may equally well be regarded as consisting of $\mathfrak{L}$, $L(\mathfrak{A})$, and the linear mapping

$$x \rightarrow L_x \tag{4}$$

of $\mathfrak{L}$ on $L(\mathfrak{A})$. Both $R(\mathfrak{A})$ and $L(\mathfrak{A})$ are linear subspaces of the total matric algebra $(\mathfrak{F})_n$ of all linear transformations on $\mathfrak{L}$.

If $\mathfrak{M}$ is a subset of $(\mathfrak{F})_n$, the algebra of all polynomials in the transformations in $\mathfrak{M}$ with coefficients in $\mathfrak{F}$ is called the *enveloping algebra* of $\mathfrak{M}$, and is denoted by $E(\mathfrak{M})$. We are particularly concerned with the enveloping algebras $E(R(\mathfrak{A}))$ and $E(L(\mathfrak{A}))$ of respectively the right and left multiplications of $\mathfrak{A}$, and with the *transformation algebra* $T(\mathfrak{A})=E(I, R(\mathfrak{A}), L(\mathfrak{A}))$ which is the algebra of all polynomials with coefficients in $\mathfrak{F}$ in the right and left multiplications of $\mathfrak{A}$ and the identity transformation I in $(\mathfrak{F})_n$. We shall have occasion to write an arbitrary element T of each of these algebras as follows:

$$T = f(R_x, R_y, \cdots) \qquad \text{for } T \text{ in } E(R(\mathfrak{A})), \tag{5}$$

$$T = f(L_x, L_y, \cdots) \qquad \text{for } T \text{ in } E(L(\mathfrak{A})), \tag{6}$$

$$T = f(I, R_x, L_x, R_y, \cdots) \qquad \text{for } T \text{ in } T(\mathfrak{A}), \tag{7}$$

where $x, y, \cdots$ are elements of $\mathfrak{A}$. In case $\mathfrak{A}$ contains a unity element 1, then $R(\mathfrak{A})$ contains I, and we may write

$$T = f(R_x, L_x, R_y, \cdots) \qquad \text{for } T \text{ in } T(\mathfrak{A}), \tag{8}$$

$x, y, \cdots$ in $\mathfrak{A}$.

If $\mathfrak{B}$ is a linear subspace of $\mathfrak{A}$, the set of all R_b for b in $\mathfrak{B}$ is a linear subspace $R(\mathfrak{B}, \mathfrak{A})$ of $R(\mathfrak{A})$, and the set of all L_b is a linear subspace $L(\mathfrak{B}, \mathfrak{A})$ of $L(\mathfrak{A})$.

An *automorphism* S of an algebra $\mathfrak{A}$ is a nonsingular linear transformation $x \rightarrow xS$ of $\mathfrak{A}$ on itself such that

$$(a \cdot x)S = aS \cdot xS \tag{9}$$

for all a, x in $\mathfrak{A}$. In terms of right and left multiplications, (9) may be written equivalently as

$$R_x S = S R_{xS} \tag{10}$$

or

$$L_x S = S L_{xS} \tag{11}$$

for all x in $\mathfrak{A}$. We shall use the facts that, if S is an automorphism of $\mathfrak{A}$, then S^{-1} is also, and if $\mathfrak{A}$ has a unity element 1, then $1S=1$.

Inasmuch as the elements T of subalgebras of $(\mathfrak{F})_n$ are themselves linear transformations, we shall denote linear transformations on subalgebras of $(\mathfrak{F})_n$—such as $T(\mathfrak{A})$, $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$—by Greek capitals, so that if Σ is a linear transformation on $T(\mathfrak{A})$, say, we may write (without confusion) the image of T under Σ as $T\Sigma$.

An automorphism S of $\mathfrak{A}$ determines an automorphism Σ of $T(\mathfrak{A})$ as follows: let T in $T(\mathfrak{A})$ be written in the form (7); then Σ is the mapping

$$T \to T\Sigma = f(I, R_{xS}, L_{xS}, R_{yS}, \cdots) = S^{-1}TS. \tag{12}$$

For if S is an automorphism of $\mathfrak{A}$, then $R_x = SR_{xS}S^{-1}$, $L_x = SL_{xS}S^{-1}$ by (10), (11), and $T = f(I, R_x, L_x, R_y, \cdots) = f(I, SR_{xS}S^{-1}, SL_{xS}S^{-1}, SR_{yS}S^{-1}, \cdots) = Sf(I, R_{xS}, L_{xS}, R_{yS}, \cdots)S^{-1} = S(T\Sigma)S^{-1}$, or $T\Sigma = S^{-1}TS$. The mapping (12) is obviously an automorphism of $T(\mathfrak{A})$. Moreover, Σ induces automorphisms (which we do not distinguish notationally from Σ) on the subalgebras $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$ of $T(\mathfrak{A})$:

$$T \to T\Sigma = f(R_{xS}, R_{yS}, \cdots), \qquad T \text{ in } E(R(\mathfrak{A})) \text{ as in (5)}, \tag{13}$$

$$T \to T\Sigma = f(L_{xS}, L_{yS}, \cdots), \qquad T \text{ in } E(L(\mathfrak{A})) \text{ as in (6)}. \tag{14}$$

If S determines Σ as in (12), then

$$R(\mathfrak{A})\Sigma = R(\mathfrak{A}), \qquad L(\mathfrak{A})\Sigma = L(\mathfrak{A}), \tag{15}$$

since $R_x\Sigma = R_{xS}$ in $R(\mathfrak{A})$ while $L_x\Sigma = L_{xS}$ in $L(\mathfrak{A})$, and the nonsingularity of Σ eliminates the possibility of proper inclusion.

2. **Automorphisms of an algebra with unity element.** Let $\mathfrak{A}$ be a non-associative algebra of order n over $\mathfrak{F}$ with unity element 1. We consider the elements of $\mathfrak{A}$ as comprising a linear space $\mathfrak{L}$ of order n over $\mathfrak{F}$. Let $\mathfrak{B}$ be any (associative) algebra of linear transformations on $\mathfrak{L}$ which contains either $R(\mathfrak{A})$ or $L(\mathfrak{A})$. We intend to reconstruct $\mathfrak{A}$ (in the sense of equivalence) as an algebra of residue classes of $\mathfrak{B}$.

Denote by $\mathfrak{N}$ the set of all transformations N in $\mathfrak{B}$ which annihilate 1, that is, for which $1N=0$. Then $\mathfrak{N}$ is a right ideal of $\mathfrak{B}$. For if N, N_1 are in $\mathfrak{N}$, then $1(\alpha N+\beta N_1) = \alpha 1N+\beta 1N_1 = 0$ for α, β in $\mathfrak{F}$, while $1NT = 0T = 0$ for any transformation T in $\mathfrak{B}$. Denote by $\mathfrak{D}$ whichever set $R(\mathfrak{A})$ or $L(\mathfrak{A})$ is assumed to be contained in $\mathfrak{B}$, and by D_x correspondingly the transformation R_x or L_x. Then $\mathfrak{B}$ is the supplementary sum $\mathfrak{B} = \mathfrak{D}+\mathfrak{N}$. For T in $\mathfrak{B}$ may be written uniquely in the form $T = D_t+N$, $1T = t$, N in $\mathfrak{N}$.

Since $1TN = tN$ which is not necessarily zero, $\mathfrak{N}$ is not in general a two-sided ideal of $\mathfrak{B}$ and we are not able to form the difference algebra $\mathfrak{B}-\mathfrak{N}$ when we take residue classes $[T]$ modulo $\mathfrak{N}$. Instead we form the difference group $\mathfrak{B}-\mathfrak{N}$ of residue classes $[T]$ modulo $\mathfrak{N}$ and have as usual a linear set over $\mathfrak{F}$ with respect to the operations $[T]+[U] = [T+U]$, $\lambda[T]=[\lambda T]$ of addition and scalar multiplication. Define multiplication in this linear set as follows:

$$[T][U] = [X] \qquad \text{where } 1X = 1T\cdot 1U, \tag{16}$$

where the multiplication on the right is that in $\mathfrak{A}$. To see that for any T, U in $\mathfrak{B}$ such an X exists, we need only to note that, if $X=D_x$ for $x=1T\cdot 1U$ in $\mathfrak{A}$, then $1X=1T\cdot 1U$. This definition of multiplication is independent of the representatives T, U since if $[T]=[T_1]$, $[U]=[U_1]$, then there exist N, N_1 in $\mathfrak{N}$ such that $T_1=T+N$, $U_1=U+N_1$, and $1T_1\cdot 1U_1=1(T+N)\cdot 1(U+N_1)=1T\cdot 1U$. With this multiplication the distributive laws hold in $\mathfrak{B}-\mathfrak{N}$. Hence $\mathfrak{B}-\mathfrak{N}$ is a non-associative algebra over $\mathfrak{F}$. Since there are no difference algebras used in this paper, there should be no confusion in the use of the notation $\mathfrak{B}-\mathfrak{N}$ for this algebra with multiplication defined by (16).

THEOREM 1. *Let $\mathfrak{A}$ be a non-associative algebra over $\mathfrak{F}$ with unity element* 1, *and $\mathfrak{B}$ be any* (*associative*) *algebra of linear transformations on $\mathfrak{A}$ containing either $R(\mathfrak{A})$ or $L(\mathfrak{A})$. If $\mathfrak{N}$ is the right ideal of transformations in $\mathfrak{B}$ annihilating* 1, *then the non-associative algebra $\mathfrak{B}-\mathfrak{N}$ with multiplication defined by* (16) *is equivalent to $\mathfrak{A}$.*

For in each residue class $[T]$ there is a unique transformation D_t in $\mathfrak{D}$ $(=R(\mathfrak{A})$ or $L(\mathfrak{A}))$ such that $1T=1D_t=t$. Then, since $\mathfrak{A}$ contains neither absolute right nor absolute left divisors of zero, the (obviously linear) mapping

$$x\rightarrow D_x\rightarrow [D_x] \tag{17}$$

is one-to-one on $\mathfrak{A}$ to $\mathfrak{B}-\mathfrak{N}$. But

$$[D_x][D_y] = [D_{xy}], \qquad x, y \text{ in } \mathfrak{A}, \tag{18}$$

since $xy=1D_x\cdot 1D_y=1D_{xy}$. Then (17) is an equivalence of $\mathfrak{A}$ and $\mathfrak{B}-\mathfrak{N}$ since $xy\rightarrow D_{xy}\rightarrow[D_{xy}]=[D_x][D_y]$ under (17).

Now $T(\mathfrak{A})$, $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$ are among the algebras of linear transformations on the vector space $\mathfrak{L}$ underlying $\mathfrak{A}$ which contain either $R(\mathfrak{A})$ or $L(\mathfrak{A})$—or both, as in the case of $T(\mathfrak{A})$—and may be used as the algebra $\mathfrak{B}$ in Theorem 1. We denote by $\mathfrak{N}_T$ the set of all N in $T(\mathfrak{A})$ annihilating 1 and write $\mathfrak{N}_R=\mathfrak{N}_T\cap E(R(\mathfrak{A}))$, $\mathfrak{N}_L=\mathfrak{N}_T\cap E(L(\mathfrak{A}))$. Then Theorem 1 implies that if multiplication in the respective algebras of residue classes is defined by (16) we have $\mathfrak{A}\cong T(\mathfrak{A})-\mathfrak{N}_T \cong E(R(\mathfrak{A}))-\mathfrak{N}_R\cong E(L(\mathfrak{A}))-\mathfrak{N}_L$.

In the proof of the next theorem we must distinguish between the cases $\mathfrak{D}=R(\mathfrak{A})$ and $\mathfrak{D}=L(\mathfrak{A})$, and we use the following equations:

$$[R_x][R_y] = [R_{xy}] = [R_xR_y], \qquad x, y \text{ in } \mathfrak{A}, \tag{19}$$

$$[L_x][L_y] = [L_{xy}] = [L_yL_x], \qquad x, y \text{ in } \mathfrak{A}, \tag{20}$$

verification of which is similar to that of (18).

THEOREM 2. *Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{N}$ be as in Theorem 1, and $\mathfrak{D}$ be $R(\mathfrak{A})$ or $L(\mathfrak{A})$, whichever is assumed to be in $\mathfrak{B}$. If Σ is an automorphism of $\mathfrak{B}$ such that $\mathfrak{N}\Sigma=\mathfrak{N}$ and $\mathfrak{D}\Sigma=\mathfrak{D}$, then Σ determines an automorphism S_Σ of $\mathfrak{A}$ as follows:*

$$S_\Sigma: \quad x \rightarrow [D_x] \rightarrow [D_x\Sigma] = [D_{x'}] \rightarrow x' = xS_\Sigma, \tag{21}$$

for x, x' in $\mathfrak{A}$, where the $[D_x]$ are elements of $\mathfrak{B}-\mathfrak{N}\cong\mathfrak{A}$.

Note first that the mapping

$$[T] \rightarrow [T\Sigma] \tag{22}$$

of $\mathfrak{B}-\mathfrak{N}$ on itself is well-defined, since if $[T]=[T_1]$ then $T=T_1+N$ for N in $\mathfrak{N}$, and $T\Sigma=(T_1+N)\Sigma=T_1\Sigma+N\Sigma=T_1\Sigma+N_1$ with N_1 in $\mathfrak{N}$ since $\mathfrak{N}\Sigma=\mathfrak{N}$. Hence $[T\Sigma]=[T_1\Sigma]$. Inasmuch as the correspondences $x\rightarrow[D_x]$ and $[D_{x'}]\rightarrow x'$ are equivalences between $\mathfrak{A}$ and $\mathfrak{B}-\mathfrak{N}$, we need only to show that (22) is an automorphism of $\mathfrak{B}-\mathfrak{N}$ in order to show that (21) is an automorphism of $\mathfrak{A}$. Now (22) is linear since $\alpha[T]+\beta[U]=[\alpha T+\beta U]\rightarrow[(\alpha T+\beta U)\Sigma]=[\alpha T\Sigma+\beta U\Sigma]=\alpha[T\Sigma]+\beta[U\Sigma]$, and is nonsingular since $[T]\rightarrow[T\Sigma]=[0]$ implies $T\Sigma=N$ in $\mathfrak{N}$, $T=N\Sigma^{-1}=N_1$ in $\mathfrak{N}$, $[T]=[0]$. Since $\mathfrak{D}\Sigma=\mathfrak{D}$, there exists an element x_1 of $\mathfrak{A}$ such that $D_x\Sigma=D_{x_1}$. But then $x_1=x'$ since there is a unique transformation in $\mathfrak{D}$ in each residue class of $\mathfrak{B}$ modulo $\mathfrak{N}$. We may write $x'=xS_\Sigma$ and

$$D_x\Sigma = D_{xS_\Sigma}. \tag{23}$$

We distinguish now between the cases $\mathfrak{D}=R(\mathfrak{A})$ and $\mathfrak{D}=L(\mathfrak{A})$. Let $\mathfrak{D}=R(\mathfrak{A})$ so that (19) holds. Then, since Σ is an automorphism of $\mathfrak{B}$, we have $[R_x][R_y]=[R_xR_y]\rightarrow[(R_xR_y)\Sigma]=[(R_x\Sigma)(R_y\Sigma)]=[R_{xS_\Sigma}R_{yS_\Sigma}]=[R_{xS_\Sigma}][R_{yS_\Sigma}]=[R_x\Sigma][R_y\Sigma]$ under (22) which is an automorphism of $\mathfrak{B}-\mathfrak{N}$ as desired. In case $\mathfrak{D}=L(\mathfrak{A})$ it follows from (20) that $[L_x][L_y]=[L_yL_x]\rightarrow[(L_yL_x)\Sigma]=[(L_y\Sigma)(L_x\Sigma)]=[L_{yS_\Sigma}L_{xS_\Sigma}]=[L_{xS_\Sigma}][L_{yS_\Sigma}]=[L_x\Sigma][L_y\Sigma]$ under (22), completing the proof of the theorem.

We shall have occasion in the proof of the next theorem to use the fact that if $\mathfrak{B}$ contains both $R(\mathfrak{A})$ and $L(\mathfrak{A})$, and if both $R(\mathfrak{A})$ and $L(\mathfrak{A})$—as well of course as $\mathfrak{N}$—are their own images under an automorphism Σ of $\mathfrak{B}$, then

$$R_x \Sigma = R_{xS_\Sigma}, \qquad L_x \Sigma = L_{xS_\Sigma} \tag{24}$$

for S_Σ defined by (21).

THEOREM 3. *Let $\mathfrak{A}$ be a non-associative algebra with unity element* 1 *and automorphism group $\mathfrak{G}$. Let $\mathfrak{H}_T$ be the group of automorphisms Σ of $T(\mathfrak{A})$ such that $\mathfrak{N}_T\Sigma = \mathfrak{N}_T$, $R(\mathfrak{A})\Sigma = R(\mathfrak{A})$, $L(\mathfrak{A})\Sigma = L(\mathfrak{A})$. Then the correspondence $S \rightarrow \Sigma$ of* (12) *is an isomorphism of $\mathfrak{G}$ onto $\mathfrak{H}_T$.*

If S is in $\mathfrak{G}$ and $S \rightarrow \Sigma$ under (12), then $1(N\Sigma) = 1S^{-1}NS = 1NS = 0S = 0$ for N in $\mathfrak{N}_T$. The nonsingularity of Σ gives $\mathfrak{N}_T\Sigma = \mathfrak{N}_T$. By (15) we have Σ in $\mathfrak{H}_T$. By Theorem 2 this Σ determines an automorphism S_Σ of $\mathfrak{A}$:

$$S_\Sigma: \quad x \rightarrow [R_x] \rightarrow [R_x \Sigma] = [R_{xS}] \rightarrow xS = xS_\Sigma$$

for all x in $\mathfrak{A}$, or $S = S_\Sigma$. Conversely, let Σ be in $\mathfrak{H}_T$. Then Σ determines an automorphism S_Σ of $\mathfrak{A}$ which in turn determines an automorphism

$$\Sigma_*: \quad T \rightarrow T\Sigma_* = S_\Sigma^{-1} T S_\Sigma, \qquad T \text{ in } T(\mathfrak{A}), \tag{25}$$

of $T(\mathfrak{A})$ by (12). Write T in the form (8). Then $T\Sigma_* = f(R_{xS_\Sigma}, L_{xS_\Sigma}, R_{yS_\Sigma}, \cdots) = f(R_x\Sigma, L_x\Sigma, R_y\Sigma, \cdots) = \{f(R_x, L_x, R_y, \cdots)\}\Sigma = T\Sigma$ by (12), (24), and the fact that Σ is an automorphism of $T(\mathfrak{A})$. That is, $\Sigma_* = \Sigma$. It is clear then that $S \rightarrow \Sigma$ is a one-to-one mapping of $\mathfrak{G}$ onto $\mathfrak{H}_T$. To see that $S \rightarrow \Sigma$ is an isomorphism we note only that if S_1, S_2 are in $\mathfrak{G}$, $S_1 \rightarrow \Sigma_1$, $S_2 \rightarrow \Sigma_2$, then for T in $T(\mathfrak{A})$ we have $T\Sigma_1 = S_1^{-1}TS_1$, $T\Sigma_1\Sigma_2 = S_2^{-1}(S_1^{-1}TS_1)S_2 = (S_1S_2)^{-1}T(S_1S_2)$, or $S_1S_2 \rightarrow \Sigma_1\Sigma_2$.

Variations in the proof of the following theorem from the proof above are trivial, consisting only of changes due to the fact that elements of $E(R(\mathfrak{A}))$ or $E(L(\mathfrak{A}))$ are generated by right or left multiplications alone.

THEOREM 4. *The correspondences $S \rightarrow \Sigma$ of* (13) *and* (14) *are isomorphisms of $\mathfrak{G}$ onto $\mathfrak{H}_R$ and $\mathfrak{H}_L$ respectively, where $\mathfrak{H}_R$ is the group of automorphisms Σ of $E(R(\mathfrak{A}))$ such that $\mathfrak{N}_R\Sigma = \mathfrak{N}_R$, $R(\mathfrak{A})\Sigma = R(\mathfrak{A})$, and $\mathfrak{H}_L$ is the group of automorphisms Σ of $E(L(\mathfrak{A}))$ such that $\mathfrak{N}_L\Sigma = \mathfrak{N}_L$, $L(\mathfrak{A})\Sigma = L(\mathfrak{A})$.*

3. **Automorphisms of an algebra without unity element.** In case we are concerned with an algebra $\mathfrak{A}_0$ of order $(n-1)$ over $\mathfrak{F}$ without a unity element, we can easily modify the results of §2 to include $\mathfrak{A}_0$. For we adjoin a unity element 1 to $\mathfrak{A}_0$ in the usual fashion to obtain an algebra $\mathfrak{A}$ of order n over $\mathfrak{F}$ containing $\mathfrak{A}_0$ (in the sense of equivalence) as an ideal. Every element x of $\mathfrak{A}$ may be written uniquely in the form

$$x = \xi 1 + x_0, \qquad \xi \text{ in } \mathfrak{F},\ x_0 \text{ in } \mathfrak{A}_0, \tag{26}$$

and if $y=\eta 1+y_0$, then $x+y=(\xi+\eta)1+(x_0+y_0)$, $\delta x=(\delta\xi)1+(\delta x_0)$ for δ in $\mathfrak{F}$, $xy=(\xi\eta)1+(\eta x_0+\xi y_0+x_0y_0)$. We shall write $\mathfrak{A}=1\mathfrak{F}+\mathfrak{A}_0$ for the algebra so defined. Any automorphism S_0 of $\mathfrak{A}_0$ may be extended in a unique fashion to an automorphism S of $\mathfrak{A}$ by defining

$$S\colon\ x \to xS = \xi 1 + x_0S_0, \tag{27}$$

x as in (26). Note that S induces the automorphism S_0 within $\mathfrak{A}_0$.

It is apparent that an automorphism S_0 of $\mathfrak{A}_0$ determines a unique automorphism Σ of $T(\mathfrak{A})$ as follows: $S_0\to S$ by (27), $S\to\Sigma$ by (12). Moreover, the linear subspaces $R(\mathfrak{A}_0, \mathfrak{A})$ and $L(\mathfrak{A}_0, \mathfrak{A})$ of $T(\mathfrak{A})$ are their own images under Σ. For if x_0 is in $\mathfrak{A}_0$, then $R_{x_0}\Sigma=R_{x_0S}=R_{x_0S_0}$ is in $R(\mathfrak{A}_0, \mathfrak{A})$ and $L_{x_0}\Sigma=L_{x_0S}=L_{x_0S_0}$ is in $L(\mathfrak{A}_0, \mathfrak{A})$.

If $\mathfrak{B}-\mathfrak{N}$ is the non-associative algebra equivalent to $\mathfrak{A}$ which was defined in §2, then $\mathfrak{A}_0$ is equivalent to the ideal $\mathfrak{C}_0$ of $\mathfrak{B}-\mathfrak{N}$ consisting of residue classes $[D_{x_0}]$ for x_0 in $\mathfrak{A}_0$, that is, for D_{x_0} in $\mathfrak{D}_0=R(\mathfrak{A}_0, \mathfrak{A})$ or $L(\mathfrak{A}_0, \mathfrak{A})$ according as $\mathfrak{D}=R(\mathfrak{A})$ or $L(\mathfrak{A})$. For, by Theorem 1, $\mathfrak{A}$ is isomorphic to $\mathfrak{B}-\mathfrak{N}$ under the mapping (17). Since $\mathfrak{A}_0$ is an ideal of $\mathfrak{A}$, the mapping

$$x_0 \to [D_{x_0}], \qquad x_0 \text{ in } \mathfrak{A}_0, \tag{28}$$

determines an ideal $\mathfrak{C}_0$ of $\mathfrak{B}-\mathfrak{N}$, and $\mathfrak{C}_0\cong\mathfrak{A}_0$.

Let Σ be an automorphism of $\mathfrak{B}$ such that $\mathfrak{N}\Sigma=\mathfrak{N}$ and $\mathfrak{D}_0\Sigma=\mathfrak{D}_0$. Then Σ determines an automorphism $S_{0\Sigma}$ of $\mathfrak{A}_0$ as follows:

$$S_{0\Sigma}\colon\ x_0 \to [D_{x_0}] \to [D_{x_0}\Sigma] = [D_{x_0'}] \to x_0' = x_0S_{0\Sigma}, \tag{29}$$

for x_0, x_0' in $\mathfrak{A}_0$. For $\mathfrak{D}=I\mathfrak{F}+\mathfrak{D}_0$, and any automorphism of $\mathfrak{B}$ leaves invariant the subspace $I\mathfrak{F}$ of order 1, so that $\mathfrak{D}\Sigma=\mathfrak{D}$. Then by Theorem 2, Σ determines an automorphism S_Σ of $\mathfrak{A}$. But S_Σ induces on $\mathfrak{A}_0$ the automorphism (29) since $D_{x_0}\Sigma=D_{x_0'}$ in $\mathfrak{D}_0$ implies x_0' is in $\mathfrak{A}_0$. Thus $x_0\to[D_{x_0}]\to[D_{x_0}\Sigma]=[D_{x_0'}]\to x_0'=x_0S_\Sigma$ is in $\mathfrak{A}_0$, or S_Σ induces $S_{0\Sigma}$ on $\mathfrak{A}_0$.

THEOREM 5. *Let $\mathfrak{A}_0$ be a non-associative algebra without unity element, and let $\mathfrak{A}=1\mathfrak{F}+\mathfrak{A}_0$. Let $\mathfrak{H}_T^0$ be the group of automorphisms Σ of $T(\mathfrak{A})$ such that $R(\mathfrak{A}_0, \mathfrak{A})\Sigma=R(\mathfrak{A}_0, \mathfrak{A})$, $L(\mathfrak{A}_0, \mathfrak{A})\Sigma=L(\mathfrak{A}_0, \mathfrak{A})$, $\mathfrak{N}_T\Sigma=\mathfrak{N}_T$. Then the correspondence $S_0\to S\to\Sigma$ of* (27) *and* (12) *is an isomorphism of the automorphism group $\mathfrak{G}_0$ of $\mathfrak{A}_0$ onto $\mathfrak{H}_T^0$.*

For if S_0 is in $\mathfrak{G}_0$, then $S_0\to S\to\Sigma$ in $\mathfrak{H}_T^0$ and $\Sigma\to S_\Sigma=S$ by Theorem 3. But then S induces the automorphism $S_{0\Sigma}$ within $\mathfrak{A}_0$. That is, $S_{0\Sigma}=S_0$. Conversely, if Σ is in $\mathfrak{H}_T^0$, then $\Sigma\to S_{0\Sigma}$ in $\mathfrak{G}_0$ by (29). But $S_{0\Sigma}\to S_\Sigma\to\Sigma_*$ by (27) and (12) and $\Sigma_*=\Sigma$ by Theorem 3. Hence the mapping $S_0 \to S \to \Sigma$ of $\mathfrak{G}_0$ on $\mathfrak{H}_T^0$ is one-to-one, and is by Theorem 3 an isomorphism.

The results analogous to Theorem 4 for algebras $\mathfrak{A}_0$ without unity quantity may be stated as follows: let $\mathfrak{H}_R^0$ be the group of automorphisms Σ of $E(R(\mathfrak{A}))$ such that $R(\mathfrak{A}_0, \mathfrak{A})\Sigma = R(\mathfrak{A}_0, \mathfrak{A})$, $\mathfrak{N}_R\Sigma = \mathfrak{N}_R$. Then the correspondence $S_0 \to S \to \Sigma$ of (27) and (13) is an isomorphism of the automorphism group $\mathfrak{G}_0$ of $\mathfrak{A}_0$ onto $\mathfrak{H}_R^0$. Let $\mathfrak{H}_L^0$ be the group of automorphisms Σ of $E(L(\mathfrak{A}))$ such that $L(\mathfrak{A}_0, \mathfrak{A})\Sigma = L(\mathfrak{A}_0, \mathfrak{A})$, $\mathfrak{N}_L\Sigma = \mathfrak{N}_L$. Then the correspondence $S_0 \to S \to \Sigma$ of (27) and (14) is an isomorphism of $\mathfrak{G}_0$ onto $\mathfrak{H}_L^0$.

4. **Inner automorphisms Σ of $T(\mathfrak{A})$.** An automorphism Σ of the associative algebra $T(\mathfrak{A})$ is called *inner* in case $T \to T\Sigma = K^{-1}TK$ for some nonsingular element K of $T(\mathfrak{A})$. We are concerned in this section with automorphisms S of $\mathfrak{A}$ which determine inner automorphisms Σ of $T(\mathfrak{A})$ under (12).

The group $\mathfrak{K}$ of all inner automorphisms of $T(\mathfrak{A})$ is an invariant subgroup of the automorphism group of $T(\mathfrak{A})$. If $\mathfrak{H}_T$ is the group of automorphisms of $T(\mathfrak{A})$ described in Theorem 3, then the intersection $\mathfrak{H}_T \cap \mathfrak{K}$ is an invariant subgroup of $\mathfrak{H}_T$. But then there is an invariant subgroup $\mathfrak{J}$ of the automorphism group $\mathfrak{G}$ of $\mathfrak{A}$ such that $\mathfrak{J} \cong \mathfrak{H}_T \cap \mathfrak{K}$ under the correspondence $S \to \Sigma$ of (12). The elements of $\mathfrak{J}$ are characterized as those automorphisms of $\mathfrak{A}$ which are themselves elements of $T(\mathfrak{A})$ by

THEOREM 6. *Let $\mathfrak{A}$ be a non-associative algebra over $\mathfrak{F}$ with unity element* 1 *and automorphism S determining an automorphism Σ of $T(\mathfrak{A})$ by* (12). *Then Σ is inner if and only if S is in $T(\mathfrak{A})$.*

If S is in $T(\mathfrak{A})$, then $T \to T\Sigma = S^{-1}TS$ is an inner automorphism of $T(\mathfrak{A})$. Conversely, if Σ is inner, there exists a nonsingular element K of $T(\mathfrak{A})$ such that $T\Sigma = K^{-1}TK$ for all T in $T(\mathfrak{A})$. In particular, $R_{xS} = R_x\Sigma = K^{-1}R_xK$. Let $1K = k$ so that $xSL_k = k \cdot xS = kR_{xS} = 1KK^{-1}R_xK = xK$ for all x in $\mathfrak{A}$, or $SL_k = K$. Since S and K are nonsingular, L_k^{-1} exists. Moreover, L_k^{-1} is in $T(\mathfrak{A})$, and $S = KL_k^{-1}$ is in $T(\mathfrak{A})$.

Perhaps it should be pointed out that Theorem 6 yields nothing in the case of central simple algebras (that is, algebras which are simple for all scalar extensions). For although it is true that, if $\mathfrak{A}$ is central simple, then $T(\mathfrak{A})$ is also and—by a well known theorem concerning associative algebras—every automorphism Σ of $T(\mathfrak{A})$ is inner, so that Theorem 6 implies that every automorphism S of $\mathfrak{A}$ is in $T(\mathfrak{A})$, it is also true [1, §8] that in this case $T(\mathfrak{A}) = (\mathfrak{F})_n$, the algebra of all linear transformations on $\mathfrak{A}$. Of course it is vacuous then to say that S is in $T(\mathfrak{A})$.

5. **The right ideals $\mathfrak{N}_T$, $\mathfrak{N}_R$, $\mathfrak{N}_L$.** We now make a more thorough analysis of the right ideals $\mathfrak{N}_T$, $\mathfrak{N}_R$, $\mathfrak{N}_L$ of $T(\mathfrak{A})$, $E(R(\mathfrak{A}))$, $E(L(\mathfrak{A}))$, respectively, and arrive in particular at criteria for the (right, left) simplicity of an algebra $\mathfrak{A}$ with unity quantity.

THEOREM 7. *An algebra $\mathfrak{A}$ with unity quantity is both commutative and associative if and only if $\mathfrak{N}_T=0$.*

For $\mathfrak{N}_T=0$ implies that $L_x-R_x=R_xR_y-R_{xy}=0$ for all x, y in $\mathfrak{A}$. That is,

$$R_x = L_x, \qquad R_xR_y = R_{xy}, \tag{30}$$

$\mathfrak{A}$ is both commutative and associative. Conversely, if (30) holds for all x, y in $\mathfrak{A}$, then T in $T(\mathfrak{A})$ has the form $T=f(R_x, L_x, R_y, \cdots)$ $=g(R_x, R_y, \cdots)=R_{g(x,y,\cdots)}$. Then $1T=0$ implies $g(x, y, \cdots)=0$ or $T=0$. Hence $\mathfrak{N}_T=0$.

The *center* $\mathfrak{Z}$ of $\mathfrak{A}$ consists of all elements c in $\mathfrak{A}$ such that

$$xc = cx, \qquad c(xy) = (cx)y = x(cy), \tag{31}$$

or equivalently

$$cL_x = cR_x, \qquad cR_{xy} = cR_xR_y = cR_yL_x \tag{32}$$

for all x, y in $\mathfrak{A}$.

THEOREM 8. *An element c is in the center $\mathfrak{Z}$ of an algebra $\mathfrak{A}$ with unity quantity if and only if $c\mathfrak{N}_T=0$.*

Certainly L_x-R_x, $R_{xy}-R_xR_y$, $R_{xy}-R_yL_x$ are in $\mathfrak{N}_T$ for all x, y in $\mathfrak{A}$. Hence if $c\mathfrak{N}_T=0$, it follows that $c(L_x-R_x)=c(R_{xy}-R_xR_y)$ $=c(R_{xy}-R_yL_x)=0$ or (32) holds, c is in the center of $\mathfrak{A}$. Conversely, if c is in the center of $\mathfrak{A}$, and if we write T in $T(\mathfrak{A})$ as in (8), it is seen by repeated application of (32) that $cT=cf(R_x, L_x, R_y, \cdots)$ $=cR_{g(x,y,\cdots)}$ where the non-associative polynomial $g(x, y, \cdots)$ $=1f(R_x, L_x, R_y, \cdots)=1T$. But if T is in $\mathfrak{N}_T$, then $1T=0$ so that $g(x, y, \cdots)=0$ and $cT=0$, $c\mathfrak{N}_T=0$.

An algebra $\mathfrak{A}$, which is not the zero algebra of order 1, is called *simple* (*right simple, left simple*) in case the only ideals (right ideals, left ideals) of $\mathfrak{A}$ are 0 and $\mathfrak{A}$.

THEOREM 9. *A non-associative algebra $\mathfrak{A}$ with unity quantity is right simple if and only if $\mathfrak{N}_R$ is a maximal proper right ideal of $E(R(\mathfrak{A}))$.*

If $\mathfrak{N}_R$ is a maximal proper right ideal of $E(R(\mathfrak{A}))$, then the only right ideal of $E(R(\mathfrak{A}))$ containing $\mathfrak{N}_R$ properly is $E(R(\mathfrak{A}))$ itself. We assume that $\mathfrak{A}$ is not right simple, so that $\mathfrak{A}$ has a right ideal $\mathfrak{Q}\neq 0$, $\mathfrak{A}$. Let $\mathfrak{P}$ be the linear set $\mathfrak{P}=R(\mathfrak{Q}, \mathfrak{A})+\mathfrak{N}_R$. Then P in $\mathfrak{P}$ has the form $P=R_q+N$, q in $\mathfrak{Q}$, N in $\mathfrak{N}_R$, and any element T of $E(R(\mathfrak{A}))$ may be written as $T=R_t+N_1$, t in $\mathfrak{A}$, N_1 in $\mathfrak{N}_R$, so that $PT=(R_q+N)(R_t+N_1)=R_qR_t+R_qN_1+NT=R_{qt}+(R_qR_t-R_{qt})+R_qN_1+NT$. Now $R_qN_1=R_a+N_2$ for a in $\mathfrak{A}$, N_2 in $\mathfrak{N}_R$, and $1R_qN_1=1R_a+1N_2$ or $a=qN_1$. Since $N_1=f(R_x, R_y, \cdots)$ while $\mathfrak{Q}$ is a right ideal of $\mathfrak{A}$, it follows that $a=qN_1=qf(R_x, R_y, \cdots)$ is in $\mathfrak{Q}$. Hence $PT=R_{qt+qN_1}+(R_qR_t-R_{qt}+N_2+NT)$ is in $\mathfrak{P}$ since $qt+qN_1$ is in $\mathfrak{Q}$ while $R_qR_t-R_{qt}+N_2+NT$ is in $\mathfrak{N}_R$. Hence $\mathfrak{P}$ is a right ideal of $E(R(\mathfrak{A}))$ containing $\mathfrak{N}_R$. Since $\mathfrak{Q}\neq 0$, $\mathfrak{A}$, it follows that $R(\mathfrak{Q}, \mathfrak{A})$, being of the same dimension over $\mathfrak{F}$ as $\mathfrak{Q}$, is neither 0 nor $R(\mathfrak{A})$, and then $\mathfrak{P}\neq\mathfrak{N}_R$, $E(R(\mathfrak{A}))$, a contradiction. Hence $\mathfrak{A}$ is right simple.

Conversely, let $\mathfrak{P}$ be any proper right ideal of $E(R(\mathfrak{A}))$ which contains $\mathfrak{N}_R$. Consider the set $\mathfrak{Q}$ of residue classes $[P]$ modulo $\mathfrak{N}_R$ for P in $\mathfrak{P}$. Then $\mathfrak{Q}$ is a linear subset of $E(R(\mathfrak{A}))-\mathfrak{N}_R\cong\mathfrak{A}$. Moreover, if $[P]$ is any element of $\mathfrak{Q}$, we write $P=R_p+N$ for p in $\mathfrak{A}$, N in $\mathfrak{N}_R$. Let $[R_t]$ be any element of $E(R(\mathfrak{A}))-\mathfrak{N}_R$. Then $PR_t=R_pR_t+NR_t=P_1$ in $\mathfrak{P}$ since $\mathfrak{P}$ is a right ideal of $E(R(\mathfrak{A}))$. Then

$$[P][R_t] = [R_p][R_t] = [R_pR_t] = [P_1] \tag{33}$$

in $\mathfrak{Q}$ by (19), and $\mathfrak{Q}$ is a right ideal of $E(R(\mathfrak{A}))-\mathfrak{N}_R\cong\mathfrak{A}$. If $\mathfrak{A}$ is right simple, then either $\mathfrak{Q}=[0]$ or $\mathfrak{Q}=E(R(\mathfrak{A}))-\mathfrak{N}_R$. In the latter case, $\mathfrak{Q}$ contains $[I]$, $\mathfrak{P}$ contains $I+N_1$ for some N_1 in $\mathfrak{N}_R$. Since $\mathfrak{P}$ also contains N_1, it follows that I is in $\mathfrak{P}$, whence $\mathfrak{P}=E(R(\mathfrak{A}))$, a contradiction. Hence $\mathfrak{Q}=[0]$, $\mathfrak{P}=\mathfrak{N}_R$, and $\mathfrak{N}_R$ is a maximal proper right ideal of $E(R(\mathfrak{A}))$.

An exactly symmetrical argument, involving left multiplications instead of right multiplications, suffices to prove

THEOREM 10. *A non-associative algebra $\mathfrak{A}$ with unity quantity is left simple if and only if $\mathfrak{N}_L$ is a maximal proper right ideal of $E(L(\mathfrak{A}))$.*

Only obvious variations on the proof above are required in the proof of

THEOREM 11. *A non-associative algebra $\mathfrak{A}$ with unity quantity is simple if and only if $\mathfrak{N}_T$ is a maximal proper right ideal of $T(\mathfrak{A})$.*

For example, to prove the converse part of the theorem, we let $\mathfrak{P}$ be any proper right ideal of $T(\mathfrak{A})$ which contains $\mathfrak{N}_T$, and let $\mathfrak{Q}$ be the linear space of residue classes $[P]$ modulo $\mathfrak{N}_T$ for P in $\mathfrak{P}$. We may write $P=R_p+N=L_p+N_0$ for N, N_0 in $\mathfrak{N}_T$, and let $[R_t]=[L_t]$ be any element of $T(\mathfrak{A})-\mathfrak{N}_T$. Then we have (33) as before, where now the quantities involved are residue classes of $T(\mathfrak{A})$ modulo $\mathfrak{N}_T$, but also we have $PL_t=L_pL_t+N_0L_t=P_2$ in $\mathfrak{P}$ so that $[L_t][P]=[L_t][L_p]=[L_pL_t]=[P_2]$ in $\mathfrak{Q}$ by (20), and $\mathfrak{Q}$ is an ideal of $T(\mathfrak{A})-\mathfrak{N}_T\cong\mathfrak{A}$. The remainder of the proof is as before.

We conclude with an analysis of the structure of the right ideal $\mathfrak{N}_T$ of $T(\mathfrak{A})$ in case $\mathfrak{A}$ of order n over $\mathfrak{F}$ (with unity quantity) is simple. In this case $T(\mathfrak{A})=(\mathfrak{Z})_s$ where the center $\mathfrak{Z}$ of $\mathfrak{A}$ is a field of degree t over $\mathfrak{F}$, and $n=st$ (see [**1**, §§8, 19]).

THEOREM 12. *Let $\mathfrak{A}$ be a simple non-associative algebra of order $n=st$ over $\mathfrak{F}$ with unity quantity and with center $\mathfrak{Z}$ of degree t over $\mathfrak{F}$. Then $\mathfrak{N}_T=\mathfrak{R}+(\mathfrak{Z})_{s-1}$, where the radical $\mathfrak{R}$ of $\mathfrak{N}_T$ has order $(s-1)$ over $\mathfrak{Z}$ and the semi-simple component of $\mathfrak{N}_T$ is the total matric algebra $(\mathfrak{Z})_{s-1}$ of degree $(s-1)$ over $\mathfrak{Z}$.*

For $\mathfrak{A}$ is central simple over $\mathfrak{Z}$. Let $(1, u_2, \cdots, u_s)$ be a fixed basis of $\mathfrak{A}$ over $\mathfrak{Z}$. Then, since $T(\mathfrak{A})=(\mathfrak{Z})_s$, it follows from Theorem 8 that $\mathfrak{N}_T$ (over $\mathfrak{Z}$) consists of all s-by-s matrices with first row zero. But the structure of this algebra of matrices, with principal idempotent

$$E=\begin{pmatrix}0 & 0\\ 0 & I_{s-1}\end{pmatrix},$$

is easily determined. Its radical $\mathfrak{R}$ consists of all matrices (with elements in $\mathfrak{Z}$) of the form

$$\begin{pmatrix}0 & 0\\ U & 0\end{pmatrix}$$

where U is any $(s-1)$-by-1 matrix. Its semi-simple component consists of all matrices (with elements in $\mathfrak{Z}$) of the form

$$\begin{pmatrix}0 & 0\\ 0 & V\end{pmatrix}$$

where V is any $(s-1)$-rowed square matrix. This is a total matric algebra $(\mathfrak{Z})_{s-1}$.

REFERENCES

1. A. A. Albert, *Non-associative algebras*. I. *Fundamental concepts and isotopy*, Ann. of Math. (2) vol. 43 (1942) pp. 685–708.

2. ———, *Quasigroups*. II. Trans. Amer. Math. Soc. vol. 55 (1944) pp. 401–419.

3. N. Jacobson, *A note on non-associative algebras*, Duke Math. J. vol. 3 (1937) pp. 544–548.

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