INNER PRODUCTS IN NORMED LINEAR SPACES

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Let T be any normed linear space [1, p. 53].[1] Then an *inner product* is defined in T if to each pair of elements x and y there is associated a real number (x, y) in such a way that $(x, y) = (y, x)$, $\|x\|^2 = (x, x)$, $(x, y+z) = (x, y) + (x, z)$, and $(tx, y) = t(x, y)$ for all real numbers t and elements x and y. An inner product can be defined in T if and only if any two-dimensional subspace is equivalent to Cartesian space [5]. A complete separable normed linear space which has an inner product and is not finite-dimensional is equivalent to (real) Hilbert space,[2] while every finite-dimensional subspace is equivalent to Euclidean space of that dimension. Any complete normed linear space T which has an inner product is characterized by its (finite or transfinite) cardinal "dimension-number" n. It is equivalent to the space of all sets $x = (x_1, x_2, \cdots)$ of n real numbers satisfying $\sum_i x_i^2 < +\infty$, where $\|x\| = (\sum_i x_i^2)^{1/2}$ [7, Theorem 32]. Various necessary and sufficient conditions for the existence of an inner product in normed linear spaces of two or more dimensions are known. Two such conditions are that $\|x+y\|^2 + \|x-y\|^2 = 2[\|x\|^2 + \|y\|^2]$ for all x and y, and that $\lim_{n\to\infty} \|x+ny\| - \|nx+y\| = 0$ whenever $\|x\| = \|y\|$ ([5] and [4, Theorem 6.3]). A characterization of inner product spaces of three or more dimensions is that there exist a projection of unit norm on each two-dimensional subspace [6, Theorem 3]. Other characterizations valid for three or more dimensions will be given here, expressed by means of orthogonality, hyperplanes, and linear functionals.

A hyperplane of a normed linear space is any closed maximal linear subset M, or any translation $x+M$ of M. A hyperplane is a supporting hyperplane of a convex body S if its distance from S is zero and it does not contain an interior point of S; it is tangent to S at x if it is the only supporting hyperplane of S containing x [8, pp. 70–74]. It will be said that an element x_0 of T is orthogonal to y ($x_0 \perp y$) if and only if $\|x_0 + ky\| \geqq \|x_0\|$ for all k, which is equivalent to requiring the existence of a nonzero linear functional f such that $f(x_0) = \|f\| \, \|x_0\|$ and $f(y) = 0$, or that $x_0 + y$ belong to a supporting hyperplane of the sphere

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[1] Numbers in brackets refer to the references at the end of the paper.

[2] "Equivalent" meaning isometric under a linear transformation [1, p. 180]. The equivalence to (real) Hilbert space follows by reasoning similar to that of [10, pp. 3–16].

$\|x\| \leqq \|x_0\|$ at the point x_0 [4, Theorem 2.1 and §5]. In a space with an inner product, $x \perp y$ if and only if $(x, y) = 0$.

Orthogonality is said to be *additive on the right* if and only if $z \perp x$ and $z \perp y$ imply $z \perp x+y$. Clearly $x \perp x$ implies $x = 0$, while $x \perp y$ implies $ax \perp by$ for any numbers a and b. Every element is orthogonal to at least one hyperplane through the origin, this hyperplane being unique for any given element if and only if: (1) For any x ($\neq 0$) and y there is a unique number a with $x \perp ax+y$; (2) The unit sphere $\|x\| \leqq 1$ of T has a tangent hyperplane at each point; (3) The norm is Gateaux differentiable; or (4) Orthogonality is additive on the right [4, Theorems 4.2, 5.1].

Orthogonality is said to be *additive on the left* if and only if $x \perp z$ and $y \perp z$ imply $x+y \perp z$. Orthogonality is not symmetric in general, and there does not necessarily exist a hyperplane orthogonal to a given element (Theorems 1 and 5). Additivity on the left does not imply strict convexity,[3] nor conversely, but a normed linear space is strictly convex if and only if: (1) For any x ($\neq 0$) and y there is a unique number a with $ax+y \perp x$; or (2) No supporting hyperplane has more than one point of contact [4, Theorems 4.3, 5.2].

Birkhoff has shown that an inner product can be defined in a normed linear space of three or more dimensions if orthogonality is symmetric and unique.[4] An equivalent condition is that $N_+(x; y) = 0$ whenever $N_+(y; x) = 0$, where $N_+(x; y) = \lim_{h \to +0}[\|x+hy\| - \|x\|]/h$ exists because of the convexity of the function $f(h) = \|x+hy\|$ [4, Theorem 6.2]. It is possible to show by a purely geometric argument that in a space of three or more dimensions orthogonality must be unique if it is symmetric, but this follows more easily from known facts about projections in normed linear spaces:

THEOREM 1. *Orthogonality is symmetric in a normed linear space T of three or more dimensions if and only if an inner product can be defined in T.*

PROOF. Let x_1 and x_2 be any two elements of a three-dimensional subspace T_0 of T. Then there is an element $y \in T_0$ orthogonal to the linear hull H_0 of x_1 and x_2 [4, Theorem 7.1]. If orthogonality is symmetric, then $H_0 \perp y$. Hence if a projection of T_0 on H_0 is defined by $z = P(z) + a_z y$, where $P(z) \in H_0$, then $\|P(z)\| \leqq \|z\|$ for all z and $\|P\| = 1$. But it is known that an inner product can be defined in a normed

[3] A normed linear space is strictly convex if $\|x+y\| = \|x\| + \|y\|$ and $y \neq 0$ imply $x = ty$ for some t.

[4] See [2]. With symmetry, uniqueness means the uniqueness for any x ($\neq 0$) and y of the number a for which $x \perp ax+y$.

linear space of three or more dimensions if there is a projection of norm one on any given closed linear subspace [6, Theorem 3]. Thus an inner product can be defined in any three-dimensional subspace of T and hence in T itself [5].

For elements x and y of a normed linear space, $x \perp y$ if and only if there is a nonzero linear functional f such that $f(x)=\|f\| \|x\|$ and $f(y)=0$, while $ax+y \perp x$ if and only if $\|kx+y\|$ is minimum for $k=a$ [4, Theorems 2.1, 2.3]. Also, the set H of all z satisfying $f(z)=\|f\|$ is a supporting hyperplane of the unit sphere at x if $f(x)=\|f\|$ and $\|x\|=1$, while any supporting hyperplane can be defined by such an equation (see Mazur [8, p. 71]). Also, H is said to be parallel to an element y if and only if $f(y)=0$ (that is, the line $\{ky\}$ does not intersect H). Interpretations of Theorem 1 by means of linear functionals and hyperplanes therefore give the following necessary and sufficient conditions for the existence of an inner product in a normed linear space of three or more dimensions:

(1) *For any elements x and y, the existence of a nonzero linear functional f with $f(x)=\|f\| \|x\|$ and $f(y)=0$ implies the existence of a nonzero linear functional g with $g(y)=\|g\| \|y\|$ and $g(x)=0$.*

(2) *For any elements x and y, $\|kx+y\|$ is minimum when $k=-f(y)/f(x)$ if f is a linear functional with $f(x)=\|f\| \|x\|$.*

(3) *The existence of a supporting hyperplane of the unit sphere at x parallel to y ($\|x\|=\|y\|=1$) implies the existence of a supporting hyperplane at y parallel to x.*

There are infinitely many different normed linear spaces of two dimensions in which orthogonality is not symmetric [2, Theorem 4]. If an isomorphism $ax+by \leftrightarrow (a, b)$ is set up between the Cartesian plane and a two-dimensional normed linear space containing x and y ($\|x\|=\|y\|=1$) and if C is the "unit pseudo-circle" of all points (a, b) for which $\|ax+by\|=1$, then orthogonality is symmetric in T if and only if the line through the origin parallel to any supporting line of C at any point p cuts C in a point at which there is a supporting line parallel to the line from p to the origin. Let B_r ($r \geqq 1$) be the normed linear space of pairs $(x_1, x_2)=x$ of real numbers, where $\|x\|^r=(|x_1|^r+|x_2|^r)$ if x_1 and x_2 are of the same sign, and $\|x\|^s=(|x_1|^s+|x_2|^s)$ otherwise, where $s=r/(r-1)$. It can easily be verified that orthogonality is symmetric in B_r for $r \geqq 1$, and that it is unique except in the limiting case $r=1$. Thus orthogonality can be symmetric and not unique in a two-dimensional space.

THEOREM 2. *An inner product can be defined in a normed linear space of three or more dimensions if and only if orthogonality is additive on the left.*

PROOF. Let T be a normed linear space of three or more dimensions, and x_1 and x_2 be any two elements. Then there are hyperplanes H_1 and H_2 with $x_1 \perp H_1$ and $x_2 \perp H_2$. Let $M = H_1 \cap H_2$. If orthogonality is additive on the left, then $ax_1 + bx_2 \perp M$ for all a and b, and any element z has a unique representation in the form $z = P(z) + y$, where $y \in M$ and $P(z) = ax_1 + bx_2$. Also, $\|z\| \geqq \|P(z)\|$ for all z, and $\|P\| = 1$. Since there is a projection of norm one on any given two-dimensional linear subspace of T, it follows as for Theorem 1 that an inner product can be defined in T [**6**, Theorem 3].

The conclusion of the above theorem is not valid without the assumption that the space be of more than two dimensions, since it is clear that for a two-dimensional normed linear space orthogonality is additive on the left if and only if for any x $(\neq 0)$ there is a unique nonzero element orthogonal to x. It therefore follows that orthogonality is additive on the left in a two-dimensional normed linear space if and only if the space is strictly convex [**4**, Theorem 4.3].

If L is a closed linear set in a Banach space B, then the *normal projection* of x on L is said to be the element u for which $x - u \perp L$, or for which $\|x-u\|$ is the distance from x to L. If L is finite-dimensional, or if the unit sphere of B is weakly compact, then normal projection is defined for all x and L [**4**, Theorem 7.2]. It was shown by Fortet [**3**, p. 45] that if orthogonality is symmetric in a uniformly convex Banach space, then normal projection is a continuous linear operation and the set H of points y with $y \perp x$ is linear and closed. However, it follows from the above theorems that H is linear for all x only if an inner product can be defined in the space R and that the existence of an inner product follows from symmetry of orthogonality. Also, $x \perp L$ if and only if there is a linear functional f with $f(x) = \|f\| \|x\|$ and $f(L) = 0$ [**4**, Theorem 2.1]. The following characterizations of inner product spaces of three or more dimensions are therefore direct consequences of Theorem 2.

(4) *The existence of a linear functional F with $F(x+y) = \|F\| \|x+y\|$ and $F(z) = 0$ whenever x, y, and z are such that there are linear functionals f and g with $f(x) = \|f\| \|x\|$, $g(y) = \|g\| \|y\|$, and $f(z) = g(z) = 0$.*

(5) *That normal projection be a linear operation.*

If a complete normed linear space has an inner product, then any linear functionals f and g can be written in the form $f(u) = (x, u)$ and $g(u) = (y, u)$, for some elements x and y [**7**, Theorem 11]. Then F of (4) can be taken as $f + g$. For any linear functional $G = Af + Bg$, there are then numbers a and b such that $G(ax + by) = \|G\| \|ax + by\|$. This condition is also sufficient for an inner product:

THEOREM 3. *An inner product can be defined in a normed linear space T of three or more dimensions if and only if it follows from $f(x)=\|f\|\,\|x\|$ and $g(y)=\|g\|\,\|y\|$ for linear fundtionals f and g and elements x and y of T that there are numbers a and b such that $f(ax+by)+g(ax+by)$ $=\|f+g\|\,\|ax+by\|$* and $ax+by\neq 0$.

PROOF. First note that if for some x there are two nonzero linear functionals F and G with $F(x)=\|F\|\,\|x\|$ and $G(x)=\|G\|\,\|x\|$, then the assumption of the theorem would imply that $|h(x)|=\|h\|\,\|x\|$ if $h=\|G\|F-\|F\|G$. But this is clearly impossible unless $h\equiv 0$, or $\|G\|F=\|F\|G$. Thus two independent linear functionals cannot take on their maximum in the unit sphere $\|x\|\leqq 1$ at the same point, which is known to imply that the unit sphere has a tangent hyperplane at each point [4, Theorem 5.1]. Now suppose that $x\perp z$ and $y\perp z$, and let T_0 be the linear hull of x, y, and z. There are then two linear functionals f and g with $f(x)=\|f\|\,\|x\|$, $g(y)=\|g\|\,\|y\|$, and $f(z)=g(z)=0$ [4, Theorem 2.1]. If x and y are not linearly independent, then $x+y\perp z$. Let x and y be linearly independent and suppose that for $u=x+y$ there are no numbers A and B satisfying $|Af(u)+Bg(u)|$ $=\|Af+Bg\|\,\|u\|$. Let C be the curve of all elements $ax+by$ with $\|ax+by\|=1$. Then there are elements x' and y' on either side of $(x+y)/\|x+y\|$ and in C for which there are linear functionals $f'=A_1f+B_1g$ and $g'=A_2f+B_2g$ with $f'(x')=\|f'\|\,\|x'\|$ and $g'(y')$ $=\|g'\|\,\|y'\|$, but such that none of the linear functionals $Af'+Bg'$ satisfy $|Af'[rx'+(1-r)y']+Bg'[rx'+(1-r)y']|=\|Af'+Bg'\|\,\|rx'+$ $(1-r)y'\|$ for any r with $0<r<1$. For each such r, there is a number α_r for which $[rx'+(1-r)y'+\alpha_r z]/\|rx'+(1-r)y'+\alpha_r z\|=v\perp z$ [4, Theorem 2.3]. If h is a linear functional defined in T_0 for which $h(v)=\|h\|\,\|v\|$ and $h(z)=0$, and if A_r and B_r are such that $A_rf'(z_0)$ $+B_rg'(z_0)=0$ for some $z_0\in T_0$ for which $h=0$ but not both f' and g' are zero, then h and $A_rf'+B_rg'$ are both zero at z_0 and z and hence are multiples of each other on T_0. Then if a_r and b_r are chosen by the assumptions of the theorem so that $\|a_rx'+b_ry'\|=1$ and $|A_rf'(a_rx'$ $+b_ry')+B_rg'(a_rx'+b_ry')|=\|A_rf'+B_rg'\|$, it follows that h is a multiple of $A_rf'+B_rg'$ and that $h(a_rx'+b_ry')|=\|h\|\,\|a_rx'+b_ry'\|$. Thus the unit sphere S contains the straight lines l_r between $a_rx'+b_ry'$ and v, since the unit sphere is convex and the tangent hyperplane defined by $h(x)=\|h\|$ contains $a_rx'+b_ry'$ and v. This tangent hyperplane at v then contains this line, but does not contain a point of C between x' and y'. But there are also tangent hyperplanes at x' and y' parallel to z, while $a_rx'+b_ry'$ is by assumption not of the form $[rx'+(1-r)y']/\|rx'$ $+(1-r)y'\|$ for any r satisfying $0<r<1$. This implies that the tangent hyperplane at v contains either x' or y' and is coincident with the tangent hyperplane at x' or y', respectively. Letting r vary from 0 to 1, it now follows from the convexity of S that the tangent hyperplanes at x' and at y' have a common point of contact and must therefore coincide, since S has a tangent hyperplane at each point. This tangent hyperplane then contains the line from x' to y', and $f'(x+y)=\|f'\| \|x+y\|$, contrary to assumption. Hence there are numbers A and B with $|Af(x+y)+Bg(x+y)|=\|Af+Bg\| \|x+y\|$. Since $Af(z)+Bg(z)=0$, this implies that $x+y \perp z$ and that orthogonality is additive on the left. It now follows from Theorem 2 that an inner product can be defined in T.

For any element x of a normed linear space there is always a hyperplane H through the origin with $x \perp H$. However, for no hyperplane H of the space[5] C of continuous functions is there an element $f \in C$ with $H \perp f$. This follows from the fact that $g \perp f$ if and only if $\min_A gf \leqq 0 \leqq \max_A gf$, where A is the set of all t with $|g(t)|=\|g\|$ [**4**, §4]. If T is one of the spaces[5] (s), (m), (c), or $l^{(p)}(p \geqq 1)$, then clearly $H \perp x$ for an infinite number of different hyperplanes H and elements x. If a normed linear space is strictly convex, then for no element x is there more than one hyperplane H with $H \perp x$, while no hyperplane is orthogonal to more than one element if the norm of T is differentiable [**4**, Theorems 4.2, 4.3]. This difference is the reason for the lack of similarity between the proofs of the following theorems.

THEOREM 4. *An inner product can be defined in a normed linear space of three or more dimensions if and only if each hyperplane through the origin is orthogonal to at least one element.*

PROOF. Let x_1 and x_2 be any two elements of a normed linear space T of three or more dimensions, and let P_0 be the linear hull of x_1 and x_2. By well-ordering the set of all linear subspaces M of T for which $P_0 \perp M$, it follows that there is a linear subspace $\overline{M}$ of T such that $P_0 \perp \overline{M}$ and $\overline{M}$ is not contained properly in any other such linear subspace. Then it is clear that $\overline{M}$ is closed. Hence if the linear hull H of P_0 and $\overline{M}$ were not T, there would be a hyperplane through the origin which contains P_0 and $\overline{M}$. If every hyperplane through the origin is orthogonal to some element, then there would be an element x such that $H \perp x$. But if $y=x_p+x_m+kx$, where $x_p \in P_0$ and $x_m \in \overline{M}$, then $\|y\| \geqq \|x_p+x_m\| \geqq \|x_p\|$, since $(x_p+x_m) \perp x$ and $x_p \perp x_m$. Thus P_0 would be orthogonal to the linear hull of $\overline{M}$ and x. Hence the linear hull of P_0 and $\overline{M}$ must be T. A projection $P(z)$ of T on P_0 can now be defined by $z=P(z)+z_m$, where $P(z) \in P_0$ and $z_m \in \overline{M}$. Since $\|P\|=1$,

[5] The notation is that of Banach [**1**, pp. 10–12].

it follows that there is a projection of unit norm on any given two-dimensional linear subspace of T and hence (as in the proof of Theorem 1) that an inner product can be defined in T.

THEOREM 5. *An inner product can be defined in a normed linear space T of three or more dimensions if and only if for any $x \in T$ there is a hyperplane H through the origin with $H \perp x$.*

PROOF. Suppose x, y, and z are any three elements of T with $x \perp z$ and $y \perp z$. If T is strictly convex, then for any u and v of T there is a unique a such that $au+v \perp u$ [4, Theorem 4.3]. Hence if H is a hyperplane through the origin with $H \perp z$, and if T is strictly convex, then $x \in H$ and $y \in H$. Thus $x+y \in H$ and $x+y \perp z$, orthogonality is additive on the left, and an inner product can be defined in T. Now suppose T is not strictly convex. Then there are elements x and y and a linear functional f with $f(x)=f(y)=\|f\|$ and $\|x\|=\|y\|=1$ [9, Theorem 6]. Let z be any other element of unit norm not in the linear set generated by x and y and let S_0 be the unit sphere of the space T_0 generated by x, y, and z. Let P_0 be the set of all points $u \in S_0$ for which $\|u\|=1$ and $f(u)=\|f\|$. Then P_0 contains the line from x to y, and is itself either a straight line segment or a section of a plane. Let L_0 be the hyperplane of T_0 with $P_0 \perp L_0$, where L_0 contains all points at which f is zero. Then for any v and each number a there is a hyperplane H_a of T_0 with $H_a \perp v+ax$. As $a \to +0$ (or as $a \to -0$), the planes H_a will have at least one limit H_+ (or H_-) in the sense that there exist sequences $\{a_i\}$ and $\{b_i\}$, with $a_i \to +0$ and $b_i \to -0$, $\lim_{a_i \to +0} \rho(w, H_{a_i})=0$ and $\lim_{b_i \to -0} \rho(w, H_{b_i})=0$, if w is any fixed element of H_+ or H_-, respectively. Since at each point of unit norm in H_a there is a supporting plane of S_0 parallel to $v+ax$, it follows that if $v \in L_0$ then neither H_+ nor H_- crosses P_0, and P_0 consists of those and only those points of the surface of S_0 in a region containing x and bounded by H_+, H_-, and the two supporting lines of P_0 parallel to v. But this is possible for arbitrary $v \in L_0$ only if P_0 is a point.

Theorems 3–5 can be given direct interpretations by means of supporting hyperplanes of the unit sphere S, as was done for Theorem 1 to get (3). The first of these interpretations can be changed somewhat to give the following nontrivial consequence of Theorem 3.

THEOREM 6. *An inner product can be defined in a Banach space if every supporting hyperplane of the unit sphere S has a point of contact and the existence of supporting hyperplanes H_1 and H_2 at points x and y of S imply that any supporting hyperplane H_3 of S satisfying $H_1 \cap H_2 \cap H_3 = 0$ have a point of contact which is in the linear hull of x and y.*

PROOF. First suppose that there is an element x and nonzero linear functionals f_1 and f_2 such that $\|x\|=1$, $f_1(x)=\|f_1\|$, and $f_2(x)=\|f_2\|$. Then x is in both of the supporting hyperplanes H_1 and H_2 of S, where H_1 and H_2 are defined by $f_1(z)=\|f_1\|$ and $f_2(z)=\|f_2\|$. If L is the set of points at which $f_1=f_2=0$, then $H_1\cap H_2=x+L$. If f_1 and f_2 are linearly independent, then the linear hull of x and L is not the whole space and there is a nonzero linear functional f_3 which is zero on x and L. Let H_3 be defined by $z\in H_3$ if and only if $f_3(z)=\|f_3\|$. Then clearly $H_1\cap H_2\cap H_3=0$. But the second hypothesis of the theorem would imply that $x\in H_3$, or $f_3(x)=\|f_3\|$, which contradicts $f_3(x)=0$. Therefore f_1 and f_2 are linearly dependent.

Now suppose that $\|x\|=\|y\|=1$, $f_1(x)=\|f_1\|$, and $f_2(y)=\|f_2\|$. If $f_3=f_1+f_2$, and H_1, H_2, H_3 are defined by $f_i(z)=\|f_i\|\,(i=1, 2, 3)$, then $x\in H_1$ and $y\in H_2$, If $H_1\cap H_2\cap H_3\neq 0$, then there exists an element w such that $f_1(w)=\|f_1\|$, $f_2(w)=\|f_2\|$, and $f_1(w)+f_2(w)=\|f_1+f_2\|$. Thus $\|f_1+f_2\|=\|f_1\|+\|f_2\|$. Since every linear functional in T takes on its maximum in the unit sphere, H_3 contains a point z of norm 1. Then $f_1(z)+f_2(z)=\|f_1+f_2\|=\|f_1\|+\|f_2\|$. Therefore $f_1(z)=\|f_1\|$ and $f_2(z)=\|f_2\|$. Hence f_1 and f_2 must be linearly dependent, and $f_1(x)+f_2(x)=\|f_1+f_2\|$. If $H_1\cap H_2\cap H_3=0$, then H_3 has a point of contact $ax+by$ ($\|ax+by\|=1$) and $f_3(ax+by)=\|f_3\|\ \|ax+by\|$. Thus it follows from Theorem 3 that an inner product can be defined.

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