A NOTE ON THE MINIMUM MODULUS OF A CLASS OF INTEGRAL FUNCTIONS

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A well known theorem due to Littlewood, Wiman, and Valiron[1] states that for any integral function of order less than one-half,

$$\log m(r) > (\text{a positive constant}) \log M(r),$$

on a sequence of circles of indefinitely increasing radius. I consider in this note a class of integral functions which have this property and prove the following theorem.

THEOREM 1. *Hypothesis*:

(1) (R_n) *is any sequence of positive numbers such that* $R_1 > 1$, $R_n/R_{n-1} \geqq \lambda > 1$.

(2) (p_n) *is any sequence of positive integers.*

(3) $a_{11}, a_{12}, \cdots, a_{1p_1}, a_{21}, \cdots, a_{2p_2}, \cdots$ *are a set of points such that* $0 < |a_{11}| \leqq |a_{12}| \leqq \cdots$ *and such that a finite number* $a_{n1}, \cdots, a_n p_n$ *lie inside the ring* $(R_n - R_n^\alpha < |z| < R_n)$ *where* $0 < \alpha < 1$.

(4) μ_n *is a sequence of positive integers such that* $\sum_1^\infty p_n/\beta^{\mu_n}$ *is convergent,* β *being any constant greater than one.*

(5) *The exponent of convergence of the points*

$$a_{nr} \exp (2\pi i v/\mu_n),$$

where $r = 1, 2, \cdots, p_n$; $v = 0, 1, 2, \cdots, \mu_n - 1$; $n = 1, 2, 3, \cdots$, is ρ $(0 \leqq \rho < \infty)$.

(6)[2] *Lower bound* $\{\mu_n\} \geqq 1 + \rho$.

Conclusion:

(7) *The canonical product*

$$f(z) = \prod_{n=1}^{\infty} \prod_{s=1}^{p_n} \left\{1 - \frac{z^{\mu_n}}{a_{ns}^{\mu_n}}\right\} \tag{8}$$

formed with these points as zeros is of order ρ; *and the values of* $r = |z|$ *for which the inequality*

$$m(r, f) > CM(r, f),$$

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[1] G. Valiron, *Lectures on the general theory of integral functions*, pp. 128–130.

[2] It is possible to choose R_n, p_n, and so on, satisfying the conditions (1) to (6). Example: $R_n = 2^{2n}$; $p_n = n^2 2^n$; $\mu_n = 2^n$. Here $\rho = 1$.

where $C=C(\lambda,\ \epsilon)>0$, *is satisfied form a set of upper density greater than* $1-1/\lambda-\epsilon$.

THEOREM 2. *If* (1), (2), (3), (4), (5), *and* (6) *hold and if* $\rho>0$ *and if further*[2]

$$\sum_{n=1}^{N}\mu_n p_n/R_N^{\rho}\to\infty \qquad \text{with } N\to\infty \tag{9}$$

then

$$\limsup_{r\to\infty}\log m(r,f)/r^{\rho}=\infty,$$

where f *is the canonical product* (8); *and the values of* r *for which* $\log m(r,\ f)>\Delta r^{\rho}$ *where* Δ *is any arbitrarily large constant form a set of upper density greater than* $1-1/\lambda-\epsilon$.

THEOREM 3. *Hypothesis: Let* $\rho>0$ *be nonintegral and* (1), (2), (3), (4), *and* (5) *hold.*[3]

Conclusion:

(10) *Any integral function of order* ρ *with exactly these zeros will be of the form*

$$F(z)=e^{g(z)}P(z)\prod_{n=n_1}^{\infty}\prod_{s=1}^{p_n}\left\{1-\frac{z^{\mu_n}}{a_{ns}^{\mu_n}}\right\}, \tag{11}$$

where $g(z)$ *is a polynomial of degree not exceeding* ρ, $P(z)$ *a polynomial;*[4] *and the values of* r *for which*

$$\log m(r,F)>(1-\epsilon)\log M(r,F)$$

holds will form a set of upper density greater than $1-1/\lambda-\epsilon$.

THEOREM 4. *If* $\rho>0$ *and* (1), (2), (3), (4), (5), *and* (9) *hold*[5] *then conclusion* (10) *holds.*

THEOREM 5. *If* (1), (2), (3), (4), (5), *and* (6) *hold and if* $m_\sigma(r)$ *and* $M_\sigma(r)$ *denote the lower and upper bounds of* $|f(z)|$, *where* $f(z)$ *is the canonical product* (8), *of order* ρ $(0\leqq\rho<\infty)$ *in the annulus* $r\leqq|z|\leqq r+r^{\sigma}$ $(\sigma<1-\rho)$ *then the values of* r *for which*[6]

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[3] For instance $R_n=2^{2n}$, $p_n=n$; $\mu_n=2^{7(n-1)}$. Here $\rho=7/2$.

[4] $P(z)$ is a polynomial having zeros at points $a_{nr}\exp(2\pi i\nu/\mu_n)$, $r=1, 2, \cdots, p_n$; $\nu=0, 1, 2, \cdots, \mu_n-1$ and $n=1, 2, \cdots, n_1-1$ only.

[5] See footnotes 2 and 3.

[6] For a number of results on the flat regions of integral functions, see J. M. Whittaker, *A property of integral functions of finite order*, Quart. J. Math. Oxford Ser. vol. 2 (1931) pp. 252–258; B. J. Maitland, *The flat regions of integral functions of finite order*, ibid. vol. 15 (1944) pp. 84–96; and the references mentioned in the paper of Maitland.

$$m_\sigma(r) > C_1 M_\sigma(r),$$

where[7] $C_1 = C_1(\lambda, \epsilon) > 0$, *holds form a set of upper density greater than* $1 - 1/\lambda - \epsilon$.

PROOF OF THEOREM 1. Let $|z| = R = \lambda^\gamma R_k$ $(0 < \gamma < 1)$, where k is so large that

$$\lambda^\gamma R_k < R_{k+1} - R^\alpha_{+1},$$

$f(z) = P_1 P_2$, where

$$P_1 = \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left\{1 - \frac{z^{\mu_n}}{a_{ns}^{\mu_n}}\right\},$$

$$P_2 = \prod_{n=k+1}^{\infty} \prod_{s=1}^{p_n} \left\{1 - \frac{z^{\mu_n}}{a_{ns}^{\mu_n}}\right\},$$

$$\begin{aligned} |P_1| &\leqq \prod_{n=1}^{k} \prod_{s=1}^{p_n} \left\{1 + \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}\right\} \\ &= \left(\prod_{n=1}^{k} \prod_{s=1}^{p_n} \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}\right)\left(\prod_{n=1}^{k} \prod_{s=1}^{p_n} \left\{1 + \frac{|a_{ns}|^{\mu_n}}{R^{u_n}}\right\}\right) \\ &= P_{11} P_{12}, \end{aligned}$$

say. Now $|a_{ns}| < R_n$,

$$|P_{12}| \leqq \prod_{1}^{k} \left\{1 + \left(\frac{R_n}{R}\right)^{\mu_n}\right\}^{p_n},$$

and $R_n/R \leqq 1/\lambda^\gamma < 1$ for $n = 1, 2, \cdots, k$, and $\sum p_n/\lambda^{\gamma\mu_n}$ is convergent. Hence

$$|P_{12}| \leqq C_2,$$

$$|P_2| \leqq \prod_{n=k+1}^{\infty} \prod_{s=1}^{p_n} \left\{1 + \frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}\right\},$$

where $|a_{ns}| \geqq |a_{k+1,s}| \geqq R_{k+1} - R^\alpha_{k+1}$,

$$\frac{R}{|a_{ns}|} \leqq \frac{R}{R_{k+1} - R^\alpha_{k+1}} \sim \frac{\lambda^\gamma R_k}{R_{k+1}} \leqq \frac{1}{\lambda^{1-\gamma}},$$

and $\sum p_n/\lambda^{(1-\gamma)\mu_n}$ is convergent. Hence

$$|P_2| \leqq C_3$$

[7] $C, C_1, C_2, \cdots$ denote finite positive (nonzero) constants.

and so

$$M(R) \leqq C_2C_3\prod_{n=1}^{k}\prod_{s=1}^{p_n}\frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}.$$

Further

$$\begin{aligned}|P_1| &= \prod_{n=1}^{k}\prod_{s=1}^{p_n}\left|1-\frac{z^{\mu_n}}{a_{ns}^{\mu_n}}\right| \\ &\geqq \prod_{n=1}^{k}\prod_{s=1}^{p_n}\left\{\frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}-1\right\} \\ &\geqq \left(\prod_{n=1}^{k}\prod_{s=1}^{p_n}\frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}\right)\left(\prod_{n=1}^{k}\prod_{s=1}^{p_n}\left\{1-\frac{|a_{ns}|^{\mu_n}}{R^{\mu_n}}\right\}\right) \\ &= P_{11}P_{14}\end{aligned}$$

say. Since $\sum p_n/\lambda^{\gamma\mu_n}$ is convergent and

$$|P_2| \geqq \prod_{n=k+1}^{\infty}\prod_{s=1}^{p_n}\left\{1-\frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}\right\} \geqq C_5,$$

$$m(R) \geqq C_4C_5\prod_{n=1}^{k}\prod_{s=1}^{p_n}\frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}\cdots, \tag{12}$$

which gives that $m(R) \geqq C_6M(R)$ where $C_6 = C_6(\lambda, \gamma)$. Now given $\epsilon > 0$ let $\epsilon_1 = \epsilon\lambda^2/(\lambda+1+\epsilon\lambda)$. Writing $\lambda^\gamma = \theta$ and $R = \theta R_k$, where $1+\epsilon_1 \leqq \theta \leqq \lambda - \epsilon_1$ and $k \geqq K$, K being so large that $R_K(\lambda-\epsilon_1) < R_{K+1} - R_{K+1}^{\alpha}$, we get $m(R) \geqq C(\lambda, \epsilon)M(R)$. This inequality holds good over a set of upper density greater than

$$\frac{(\lambda-\epsilon_1)-(1+\epsilon_1)}{\lambda-\epsilon_1} = 1 - \frac{1}{\lambda} - \epsilon.$$

PROOF OF THEOREM 2. We know from (12) that $m(R, f) \geqq C_4C_5X$, where

$$X = \prod_{n=1}^{k}\prod_{s=1}^{p_n}\frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}} \geqq \lambda^{(\gamma\sum_1^k \mu_n p_n)}$$

$$\begin{aligned}\log m(R, f) &\geqq \log (C_4C_5) + \log X \geqq \log (C_4C_5) + \gamma \log \lambda\left(\sum_1^k \mu_n p_n\right) \\ &> \Delta R^\rho \qquad \text{for all large } R.\end{aligned}$$

Hence $\limsup_{r\to\infty} \log m(r, f)/r^\rho = \infty$. Further, the values of r for which $\log m(r, f) > \Delta r^\rho$ form a set of upper density greater than $1 - 1/\lambda - \epsilon$.

PROOF OF THEOREM 3. Given $\epsilon>0$, let $\epsilon_2=\epsilon/(2-\epsilon)$. Since

$$\sum \mu_n p_n/(R_n - R_n^{\alpha})^{\rho-\epsilon_3}$$

is divergent we have

$$\mu_n p_n \geqq R_n^{\rho-\epsilon_4} \qquad \text{or } n = k_1, k_2.$$

Let $|z|=R=\lambda^\gamma R_k$ ($0<\gamma<1$ and $1+\epsilon_1\leqq\lambda^\gamma\leqq\lambda-\epsilon_1$), where k takes the values $k_1, k_2, \cdots$. If $X = \prod_{n=n_1}^k\prod_{s=1}^{p_n} R^{\mu_n}/|a_{ns}|^{\mu_n}$ then $X\geqq\exp\{\gamma\log\lambda\sum_{n_1}^k\mu_n p_n\}$ and so $\log X\geqq C_6\sum_{n_1}^k\mu_n p_n\geqq C_6 R_k^{\rho-\epsilon_4}=C_7R^{\rho-\epsilon_4}$. Choosing k and hence R sufficiently large we have, as in Theorem 1,

$$m(R, F) > C_8 \exp\{\log X - C_9 R^{[\rho]}\},$$

$$\log m(R, F) > \log C_8 + \log X - C_9 R^{[\rho]}$$

$$> (1-\epsilon_2)\log X.$$

Similarly $\log M(R, F)<(1+\epsilon_2)\log X$ which gives

$$\frac{\log m(R, F)}{\log M(R, F)} > \frac{1-\epsilon_2}{1+\epsilon_2} = 1-\epsilon.$$

As in Theorem 1, this result holds for values of R forming a set of upper density greater than $1-1/\lambda-\epsilon$.

Theorem 4 can be similarly proved.

PROOF OF THEOREM 5. We know that for $|z|=R=\lambda^\gamma R_k$ ($0<\gamma<1$, $1+\epsilon_1\leqq\lambda^\gamma\leqq\lambda-\epsilon_1$)

$$m(R, f) \geqq C_4C_5\prod_{n=1}^{k}\prod_{s=1}^{p_n}\frac{R^{\mu_n}}{|a_{ns}|^{\mu_n}}.$$

We can choose k so large that $R'=R+R^\sigma<\lambda^{\gamma+\epsilon_5}R_k$, where $\gamma+\epsilon_5<1$,

$$R' < R_{k+1} - R_{k+1}^{\alpha}.$$

Now

$$M(R', f) < C_{10}\prod_{n=1}^{k}\prod_{s=1}^{p_n}\frac{R'^{\mu_n}}{|a_{ns}|^{\mu_n}}$$

and therefore

$$\frac{m(R, f)}{M(R', f)} > \frac{C_4C_5}{C_{10}}\left(\frac{R}{R'}\right)^{\sum_{n=1}^k\mu_n p_n}.$$

Now $Y=(R'/R)^{-\sum_1^k\mu_n p_n} = (1+R^{\sigma-1})^{-\sum_1^k\mu_n p_n}$. Further $\sum_1^k\mu_n p_n<(C_{11}\log R)R^{\rho+\epsilon_6}<R^{\rho+\epsilon_7}$ for all large R. Hence $Y>\exp\{-R^{\rho+\epsilon_7}$

$\log(1+R^{\sigma-1})\}$ and $R^{\rho+\epsilon_7}\log(1+R^{\sigma-1})\sim R^{\rho+\epsilon_7+\sigma-1}\to 0$ as $R\to\infty$, since $\sigma<1-\rho$ and ϵ_7 can be chosen so small that $\sigma<1-\rho-\epsilon_7$. Hence $Y>1/2$ for all large R and so

$$\frac{m(R,f)}{M(R',f)} > \frac{C_4C_5}{2C_{10}}.$$

Further

$$\frac{m(R',f)}{M(R',f)} > C_{11}.$$

Hence

$$\frac{m_\sigma(R)}{M_\sigma(R)} = \min\left\{\frac{m(R)}{M(R')},\ \frac{m(R')}{M(R')}\right\} \geqq \min\left\{\frac{C_4C_5}{2C_{10}},\ C_{11}\right\} \geqq C_1.$$

The values of R for which this result holds form a set of upper density greater than $1-1/\lambda-\epsilon$.

Added in proof. The positive numbers ϵ and ϵ_4 are chosen so small that

$$1/\lambda + \epsilon < 1; \qquad [\rho] + \epsilon_4 < \rho.$$

In the proof of Theorem 1 we showed that

$$M(R) \leqq C_2C_3P_{11}; \qquad m(R) \geqq C_4C_5P_{11},$$

both relations holding for all R such that

$$(1+\epsilon_1)R_k \leqq R \leqq (\lambda-\epsilon_1)R_k \qquad (k>K).$$

Here

$$C_2 = \prod_{n=1}^{\infty}\left\{1+\left(\frac{1}{1+\epsilon_1}\right)^{\mu_n}\right\}^{p_n},\quad C_4 = \prod_{n=1}^{\infty}\left\{1-\left(\frac{1}{1+\epsilon_1}\right)^{\mu_n}\right\}^{p_n},$$

$$C_3 = \prod_{n=1}^{\infty}\left\{1+\left(1-\frac{\epsilon_1}{2\lambda}\right)^{\mu_n}\right\}^{p_n},\quad C_5 = \prod_{n=1}^{\infty}\left\{1-\left(1-\frac{\epsilon_1}{2\lambda}\right)^{\mu_n}\right\}^{p_n}.$$

If $C=C_4C_5/C_2C_3$ we have

$$m(R) \geqq CM(R),$$

the inequality holding over a set of upper density greater than $1-1/\lambda-\epsilon$. If we further suppose that $\lambda=R_n/R_{n-1}$ $(n=2, 3, \cdots)$, then this inequality holds good over a set of upper density greater than $1-\lambda\epsilon(1+\epsilon)/(\lambda-1)$.

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