NOTE ON THE DERIVATIVES OF FUNCTIONS ANALYTIC IN THE UNIT CIRCLE

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In a recent paper,[1] W. Seidel and the present writer established distortion theorems for various classes of functions analytic in the unit circle; more explicitly, established relations between the derivatives of functions and their radii of univalence and of p-valence, with particular reference to behavior as a point approaches the circumference. Classes studied in detail were functions respectively univalent, bounded, omitting two values, p-valent, and having a bounded radius of univalence. The last-named class clearly includes the class of functions $f(z)$ each analytic in $|z| < 1$ and transforming $|z| < 1$ onto a Riemann configuration of finite area. It is the primary object of the present note to study this included class to the same end more effectively by other methods. Terminology and notation are uniform with the paper referred to; unless otherwise specified, references in the present note are to that paper; we shall also have occasion to use the results of a subsequent paper by Loomis.[2]

We denote by K_M the class of functions $f(z)$ analytic in $|z| < 1$ and transforming the region $|z| < 1$ into a region whose area (counted according to the multiplicity of covering) is not greater than πM^2:[3]

$$(1) \qquad \iint_{|z|<1} |f'(z)|^2 dS. \leqq \pi M^2.$$

If we set $f(z) \equiv \sum_0^\infty a_n z^n$, inequality (1) becomes

$$(2) \qquad \sum_1^\infty n |a_n|^2 \leqq M^2,$$

whence we have

$$(3) \qquad |a_n| \leqq \frac{M}{n^{1/2}}.$$

We proceed now to establish the following theorem.

THEOREM 1. *If $f(z)$ is of class K_M, with $w_0 = f(z_0)$, we have*

(4) $D_1(w_0) \leq |f'(z_0)| (1 - |z_0|^2) \leq [6MD_1(w_0)]^{1/2}.$

The first inequality in (4) has already been established (loc. cit. §4). We proceed to prove the second inequality in (4).

The proof is by the method of Landau employed in the previous paper (§18). Let us assume first $f(0) = 0$, $f'(0) = 1$, from which it follows by (3) that the series $\sum a_n z^n$ is term by term not greater in absolute value than the series

(5) $z + Mz^2 + Mz^3 + Mz^4 + \cdots ;$

the latter series has the sum ($|z| < 1$)

(6) $z + \dfrac{Mz^2}{1 - z}.$

Of course we have $M \geq 1$.

For the value $r = 1/(4M)$ we have

(7) $r - \max_{|z|=r} |f(z) - z| \geq r - \dfrac{Mr^2}{1 - r} \geq r - \dfrac{4Mr^2}{3} = \dfrac{1}{6M} = \phi(M).$

Thus on the circle $|z| = r$ we have

$$\left| \frac{f(z) - z}{z - w} \right| < 1$$

for $|w| < \phi(M)$, so by Rouché's theorem $f(z)$ takes on the value w precisely as many times in $|z| < r$ as does the function z, namely once. That is to say, the map $w = f(z)$, $|z| < 1/(4M)$, covers smoothly the circle $|w| < \phi(M) = 1/(6M)$. Thus we have

(8) $D_1(0) \geq 1/6M.$

Given an arbitrary z_0, with $|z_0| < 1$, the case $f'(z_0) = 0$ trivially implies (4), with each member zero; in the case $f'(z_0) \neq 0$ we construct the frequently used auxiliary function

(9) $\psi(z) = \dfrac{f((z + z_0)/(1 + \bar{z}_0 z)) - f(z_0)}{f'(z_0)(1 - |z_0|^2)}.$

We have $\psi(0) = 0$, $\psi'(0) = 1$. Since the transformation $z' = (z + z_0)/(1 + \bar{z}_0 z)$ maps $|z| < 1$ smoothly onto $|z'| < 1$, it is clear that the Riemann configuration which is the image of $|z| < 1$ under

the transformation $w = f(z')$ is the same as the image of $|z| < 1$ under the transformation $w = f(z)$; consequently $\psi(z)$ is of class $K_{M'}$, with

$$M' = \frac{M}{|f'(z_0)|\,(1 - |z_0|^2)}.$$

By inequality (8) we may now write

$$D_1(w_0) \geqq \frac{|f'(z_0)|\,(1 - |z_0|^2)}{6M},$$

where $D_1(w_0)$ here refers to the image of $|z| < 1$ under the transformation $w = \psi(z)$. By equation (9) we obviously have

$$D_1(w_0)\big|_{w=\psi(z)} = D_1(w_0)\big|_{w=f(z)} \div |f'(z_0)|\,(1 - |z_0|^2),$$

which implies the second inequality in (4) and completes the proof of Theorem 1.

The order $1/2$ as $D_1(w_0)$ in (4) approaches zero cannot be improved; compare loc. cit. §10.

This method just used yields a slightly more favorable inequality than (4). Series (5) can be replaced by

$$(10) \qquad z + \frac{Mz^2}{2^{1/2}} + \frac{Mz^3}{2^{1/2}} + \frac{Mz^4}{2^{1/2}} + \cdots = z + \frac{Mz^2}{2^{1/2}(1-z)}.$$

Then the latter part of (7) can be replaced by

$$r - \frac{Mr^2}{2^{1/2}(1-r)} \geqq r - \frac{4Mr^2}{3 \cdot 2^{1/2}} = \frac{6 - 2^{1/2}}{24M} = \phi(M).$$

The second member of (8) is $\phi(M)$, so in (4) the number 6 may be replaced by

$$\frac{24}{6 - 2^{1/2}} = \frac{72 + 12 \cdot 2^{1/2}}{17} = 5.24.$$

We remark too that in (10) the number M can be replaced by $(M^2 - 1)^{1/2}$ with a slight improvement in (4).

In extending Theorem 1 to higher derivatives, we shall prove the following theorem.

THEOREM 2. *If the function $f(z) = \sum_0^\infty a_n z^n$ is of class K_M, then we have*

$$(11) \qquad D_p(0) \leqq \sum_1^p |a_n| \leqq 2^p \cdot A_p (M \log 2)^{p/(p+1)} [D_p(0)]^{1/(p+1)},$$

$$A_p = (p+1)[(p+1)/p]^{p/(p+1)}(4p)^p.$$

The first inequality in (11) is established by Loomis (loc. cit.), a sharpening of an inequality due to Seidel and Walsh. To establish the second inequality we write for $|z_0| < 1$, with the double integral taken over $|z - z_0| < 1 - |z_0|$,

$$(12) \quad |f'(z_0)|^2 = \frac{1}{\pi(1 - |z_0|)^2} \left| \int \int [f'(z)]^2 dS \right| \leqq \frac{M^2}{(1 - |z_0|)^2} .$$

If we assume $f(0) = 0$, which here involves no loss of generality, we have

$$(13) \quad |f(z)| = \left| \int_0^z f'(t) dt \right| \leqq \int_0^r \frac{M \, dt}{1 - t} = - M \log (1 - r),$$

$$|z| = r < 1.$$

The function $F(z) \equiv f(z/2) \equiv a_1(z/2) + a_2(z/2)^2 + \cdots$ is analytic in the circle $|z| < 1$ and by (13) is of modulus not greater than $M \log 2$ there. Consequently we have (Loomis, loc. cit.)

$$(14) \quad \frac{|a_1|}{2} + \frac{|a_2|}{2^2} + \cdots + \frac{|a_p|}{2^p} \leqq A_p(M \log 2)^{p/(p+1)} [D_p(0)]^{1/(p+1)},$$

where $D_p(0)$ refers to the largest m-sheeted circle ($m \leqq p$) with center O and contained in the image of $|z| < 1$ under the transformation $w = F(z)$. Inequality (14), valid where $D_p(0)$ refers to the image of $|z| < 1/2$ under the transformation $w = f(z)$, is a fortiori valid if $D_p(0)$ refers to the image of $|z| < 1$ under the transformation $w = f(z)$, and (11) follows at once.

If inequality (11) is rewritten so as to apply to the function

$$g(\zeta) = f\left(\frac{\zeta + z}{1 + \zeta \bar{z}}\right), \qquad |z| < 1, \ |\zeta| < 1,$$

it appears (Siedel and Walsh, loc. cit. Chapter I, Lemma 2) that $D_p(w_n) \to 0$ is a necessary and sufficient condition for the relations

$$f^{(m)}(z_n)(1 - |z_n|^2)^m \to 0, \qquad m = 1, 2, \cdots, p.$$

For functions $f(z)$ of class K_M, this remark establishes the relation $D_p(w_n) \to 0$ as a consequence of $w_n = f(z_n)$, $|z_n| \to 1$, by virtue of the following theorem:

THEOREM 3. *If $f(z)$ belongs to K_M, then we have for every k*

$$(15) \qquad \lim_{|z| \to 1} f^{(k)}(z)(1 - |z|^2)^k = 0;$$

the approach is uniform in the sense that if $\epsilon > 0$ is given, there exists δ_ϵ such that $1 - |z| < \delta_\epsilon$ implies $|f^{(k)}(z)(1 - |z|^2)^k| < \epsilon$.

Of course the uniformity of the approach is a consequence of the existence of the limit. We introduce the notation

$$(16) \qquad \iint_{r < |z| < 1} |f'(z)|^2 dS = \Phi(r),$$

whence $\lim_{r \to 1} \Phi(r) = 0$ monotonically. If $\rho \, (> 1/2)$ is arbitrary, $\rho < |z_0| < 1$, we choose $r \le 2\rho - 1$, and the integral in (12) is in absolute value not greater than $\Phi(r)$, so we have

$$|f'(z_0)|^2 \le \frac{\Phi(r)}{\pi(1 - |z_0|)^2},$$

from which (15) follows uniformly, for the value $k = 1$.

We now take $|z| > 1/3$, $\rho = (1 - |z|)/2$, and use Cauchy's integral formula

$$f'(z) = \frac{1}{2\pi i} \int_{|t-z|=\rho} \frac{f'(t)dt}{t - z}.$$

Successive differentiation yields

$$f^{(k)}(z) = \frac{(k - 1)!}{2\pi i} \int_{|t-z|=\rho} \frac{f'(t)dt}{(t - z)^k},$$

whence we have

$$|f^{(k)}(z)| \le (k - 1)![\max |f'(t)|, \text{ for } |t - z| = \rho]/\rho^{k-1},$$

and by the definition of ρ,

$$(1 - |z|)^{k-1}|f^{(k)}(z)| \le 2^{k-1} \cdot (k - 1)![\max |f'(t)|, \text{ for } |t - z| = \rho].$$

We have further $(1 - |t|) \ge (1 - |z|)/2$, so we deduce

$$(1 - |z|)^k |f^{(k)}(z)|$$
$$\le 2^{k-1}(k - 1)!(1 - |z|)[\max |f'(t)|, \text{ for } |t - z| = \rho]$$
$$\le 2^k(k - 1)![\max |f'(t)| (1 - |t|), \text{ for } |t - z| = \rho],$$

which implies (15) uniformly and establishes Theorem 3.

It may be noted that we have derived (15) for $k > 1$ merely from (15) for $k = 1$ as a consequence only of the analyticity of $f(z)$ in $|z| < 1$.

For the class K_M we can make no deductions regarding the rapidity of approach to zero in (15):

THEOREM 4. *Let the function $Q(r)$ be defined and positive for $0 < r < 1$, with $\lim_{r \to 1} Q(r) = 0$. Let the positive integer m and the positive number M be given. Then there exists a function $f(z)$ of class K_M, continuous and schlicht for $|z| \leq 1$, and there exists a sequence of points z_k with $0 < z_k < 1$, $z_k \to 1$, such that we have*

$$\lim_{k \to \infty} \frac{f^{(m)}(z_k)(1 - |z_k|^2)^m}{Q(|z_k|)} = \infty.$$

It suffices to choose here a suitable constant multiple of the function $f(z)$ previously constructed (loc. cit. §9).

It is clear from Theorem 1 that the two conditions

(17) $D_1(w_n) \to 0, \qquad w_n = f(z_n),$

(18) $f'(z_n)(1 - |z_n|^2) \to 0,$

for a function $f(z)$ of class K_M are equivalent in the sense that each implies the other. If z_n is a sequence ($|z_n| < 1$) which has no limit point in a zero of $f'(z)$ interior to $|z| = 1$, it is clear that (18) implies

(19) $|z_n| \to 1;$

reciprocally (19) implies (18) by virtue of Theorem 3. That is to say, for a function of class K_M these three conditions are all equivalent, except that (17) and (18) are trivially fulfilled for any sequence or subsequence z_n which approaches a zero of $f'(z)$ interior to $|z| = 1$.[4]

Theorem 3 asserts that if $f(z)$ belongs to K_M, then (19) implies (15) for every k. We have remarked that Theorem 2 as applied to the function

$$f\left(\frac{\zeta + z}{1 + \bar{z}\zeta}\right)$$

shows that

(20) $\lim_{n \to \infty} D_p(w_n) = 0$

is equivalent to (15) for $k = 1, 2, \cdots, p$. Thus for a function $f(z)$ of class K_M, the three relations (19), (20), and (15) for $k = 1, 2, \cdots, p$ are all equivalent except that the latter two relations are trivially fulfilled for any sequence or subsequence z_n which approaches a common zero of $f^{(k)}(z)$, $k = 1, 2, \cdots, p$, interior to $|z| = 1$.

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[4] In the study of the equivalence of the relations (17), (18), and (19) for bounded univalent functions (loc. cit. p. 211), the phrase "every point of R_{N_δ}" should read "every point of R exterior to R_{N_δ}."

In considering the relationships between (17), (18), and (19), it is of interest to study the geometric significance of (17) independently of the class K_M. We shall prove:

THEOREM 5. *Let the function $w=f(z)$ be analytic interior to $|z|<1$, and map $|z|<1$ onto the Riemann configuration R. A necessary and sufficient condition that (19) imply (17) with $w_n=f(z_n)$ is that there exist no $\delta\ (>0)$ to which correspond an infinity of mutually nonoverlapping smooth circles $C_1, C_2, \cdots$ of radius δ in R.*

If for some $\delta\ (>0)$ such circles C_n do exist, their respective centers w_n (considered as points of R, not merely as values of the complex variable) have no limit point in R, and the corresponding points z_n have no limit point interior to $|z|<1$; thus (19) is satisfied but not (17), so (19) does not imply (17). Conversely, let us assume that (19) does not imply (17); we suppose (19) satisfied for a particular sequence z_n, but suppose that (17) is not satisfied; we shall show the existence of a suitable $\delta\ (>0)$ and corresponding circles C_n in R. Choose a subsequence of the z_n so that for a suitably chosen $\delta\ (>0)$ the values $D_1(w_n)$ are all greater than or equal to 2δ; we change the notation if necessary so as to have $D_1(w_n)\geqq 2\delta$ for every n. If any subsequence Γ_{n_k} of these circles Γ_n (with respective centers w_n and radii $D_1(w_n)$) have centers w_{n_k} in R which approach as limit a point w_0 of R, then (loc. cit. §15) by the continuity of $D_1(w)$ as a function of w, we have $D_1(w_0)\geqq 2\delta$; consequently we have $w_{n_k}\rightarrow w_0$, $z_{n_k}\rightarrow z_0$ with $|z_0|<1$, contrary to hypothesis. At most a finite number of the points $w_2, w_3, \cdots$ lie interior to the circle γ_1 whose center is w_1 and radius 2δ; suppose none of the points w_n for $n\geqq N_1$ lies interior to γ_1. At most a finite number of the points w_n for $n>N_1$ lie interior to the circle γ_2 whose center is w_{N_1} and radius 2δ; suppose none of the points w_n for $n\geqq N_2\ (>N_1)$ lies interior to γ_2. Let γ_3 be the circle whose center is w_{N_2} and radius 2δ. We continue this process indefinitely, and find a sequence of circles $\gamma_1, \gamma_2, \cdots$ all of radius 2δ; the center of γ_k cannot lie interior to any of the circles $\gamma_1, \gamma_2, \cdots, \gamma_{k-1}$. The circles C_k concentric with the circles γ_k and having the common radius δ lie in R and no two of them overlap. The proof is complete.

The condition of Theorem 5 is equivalent to the condition that *there exist no $\epsilon\ (>0)$ such that for every $n=1, 2, 3, \cdots$, the image R_n of the annulus $1>|z|>1-1/n$ under the transformation $w=f(z)$ contains a smooth circle of radius ϵ.* Let each R_n contain a smooth circle Γ_n of radius ϵ; then R_n contains the concentric circle C_n of radius $\delta=\epsilon/2$. Points of each C_n lie in but a finite number of regions R_k, so there exist among the C_n an infinite number of nonoverlapping

smooth circles in R, each of radius δ. Conversely, let there exist an infinity of nonoverlapping smooth circles Γ_n of radius δ (>0) in R; let C_n denote the concentric circles of radius $\epsilon = \delta/2$, and w_n denote the respective centers. Any closed region $R - R_k$ contains points belonging to at most a finite number of the circles C_n, for otherwise there exists a point z_0 with $|z_0| \leqq 1 - 1/k$ such that every neighborhood of z_0 contains points z corresponding to points w of an infinity of those circles C_n; every neighborhood in R of the point $w_0 = f(z_0)$ then contains points of an infinity of the circles C_n, which is impossible. Consequently each R_k contains at least one C_n; the equivalence is established.

As a consequence of Theorem 5 we prove:

COROLLARY 1. *Let the function $w = f(z)$ be analytic interior to $|z| < 1$, and map $|z| < 1$ onto the Riemann configuration R. A necessary and sufficient condition that* (19) *imply* (20) *for each p is that there exist no δ (>0) to which correspond an infinity of mutually nonoverlapping smooth circles of radius δ in R.*

We have the inequality

$$D_1(w) \leqq D_p(w),$$

so if (19) implies (20) for a single value of p, then (19) also implies (17), and by Theorem 5 there exists no δ (>0) to which correspond an infinity of mutually nonoverlapping smooth circles of radius δ in R. Conversely, for given p suppose (19) does not imply (20); we assume (19) satisfied for a particular sequence z_n but assume (20) for a particular p not satisfied; we shall show the existence of a suitable δ and the corresponding circles. By the method of proof of Theorem 5 there exists a δ (>0) and a sequence of m-sheeted circles ($m \leqq p$) Γ_1, Γ_2, $\cdots$, of radius not less than 2δ, which belong to R and are mutually nonoverlapping. Each circle Γ_j contains at least one smooth circle C_j of radius δ;[5] for an m-sheeted circle has branch points of total multiplicity $m - 1$; a branch point of order $m - 1$ lies in m sheets, a branch point of order $m - 2$ lies in $m - 1$ sheets, $\cdots$, a branch point of order 1 lies in 2 sheets; at least two sheets of Γ_j contain but one branch point; and each of these two sheets contains a smooth circle of radius δ. The circles C_j are mutually nonoverlapping, so Corollary 1 is established. If for any function analytic in $|z| < 1$ condition (19) implies (17), then (19) also implies (20) for every p.

The condition of Theorem 5 is obviously satisfied by any function

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[5] This fact follows also from a lemma due to Grünwald and Turán as sharpened by Grünwald and Vázsonyi; see Acta Univ. Szeged. vol. 8 (1936–1937) pp. 236–240.

of class K_M, and by functions bounded or unbounded of many other types. Moreover we prove the following corollary.

COROLLARY 2. *If $f(z)$ is analytic and schlicht in $|z| < 1$, and maps $|z| < 1$ onto a region R, then a necessary and sufficient condition for the equivalence of (17), (18), and (19) is that there exist no δ (>0) to which correspond an infinity of mutually nonoverlapping smooth circles C_n of radius δ in R.*

We have (loc. cit. §4) the inequalities

$$(21) \qquad D_1(w_0) \leqq |f'(z_0)| (1 - |z_0|^2) \leqq 4D_1(w_0),$$

from which the equivalence of (17) and (18) follows. The derivative $f'(z)$ is different from zero at all points in $|z| < 1$, so (18) must imply (19). Corollary 2 now follows from Theorem 5.

For an *arbitrary* function $f(z)$ analytic in $|z| < 1$, the existence of $\delta(>0)$ and corresponding mutually nonoverlapping smooth circles C_j in R of radius δ implies

$$\limsup_{|z| \to 1} |f'(z)| (1 - |z|^2) > 0,$$

by the first of inequalities (21); see loc. cit. §4.

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