186. Morgan Ward: *Elliptic divisibility sequences.*

The author obtains all solutions of the difference equation $\omega_{m+n}\omega_{m-n} = \omega_{m+1}\omega_{m-1}\omega_n^2 - \omega_{n+1}\omega_{n-1}\omega_m^2$ over the field of rationals, and studies the arithmetical properties of integral solutions. These solutions include the well known Lucas function $u_n = (a^n - b^n)/(a-b)$ which satisfies a linear difference equation of order two; the more general numerical functions here investigated have very similar arithmetical properties. The conjecture of Lucas that there exists a connection between solutions of a linear difference equation of order three or four and elliptic functions is shown to be false so far as the difference equation studied is concerned despite Lucas' explicit assertion to the contrary (Amer. J. Math. vol. 1 (1878) p. 203.) (Received March 8, 1947.)

187. N. A. Wiegmann: *Some theorems on normal matrices with analogs of the generalized principal axis transformation.*

The following are among a number of theorems obtained on normal matrices: A necessary and sufficient condition that a matrix be normal is that its polar matrices be expressible as polynomials in the matrix; if A, B, and AB are normal, then BA is a normal matrix; a necessary and sufficient condition that the product AB of two normal matrices be normal is that each commute with the hermitian polar matrix of the other; the product AB of any two square matrices is normal if and only if there exist a unitary U and a nonsingular P such that $UAP = D$ and $P^{-1}BU^{CT} = T$ where D and T are certain diagonal and triangle matrices, respectively. Several analogs of the generalized principal axis transformation for a set of matrices A_α are shown to hold. (Received March 21, 1947.)

188. R. L. Wilson: *A finite method for the determination of the Galois group of an equation with an application to the problem of reducibility.*

This paper gives a sieve process for the determination of the Galois group, which depends only upon the ability to obtain the rational roots of a system of induced equations. If the given equation is of degree n, the Galois group G must be contained in the symmetric group of degree n. Hence, it is only necessary to ascertain which, if any, of the subgroups of the symmetric group contain G. If Γ is any subgroup of the symmetric group, the author has shown how to construct a function, ϕ, of the roots of the given equation which is invariant under precisely the permutations of Γ. The induced equation is then formed, having for its roots the function ϕ together with its conjugate functions. The theorem is then established that $G \subseteq \Gamma$ if and only if the induced equation has at least one rational root. By a different choice of the function ϕ, this same method is applied to the problem of the reducibility of a polynomial. (Received March 15, 1947.)

Analysis

189. J. E. Bearman: *Rotations in the product of two Wiener spaces.* Preliminary report.

Let $X(t)$ and $Y(t)$ be elements of C, the space of all continuous functions on $0 \leqq t \leqq 1$ which vanish at $t=0$. Then if $F[X, Y]$ is a Wiener summable functional over $C \otimes C$, it is shown that the "rotation" $X(t) = x(t)\cos\theta - y(t)\sin\theta$, $Y(t) = x(t)\sin\theta + y(t)\cos\theta$

preserves measurability and measure, and that $\int_c^w \int_c^w F[X, Y] d_w X d_w Y = \int_c^w \int_c^w F[x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta] d_w x d_w y$. (Received March 19, 1947.)

190. Richard Bellman: *On the solutions of the heat equation.* Preliminary report.

It is proved that if u is a solution of the heat equation, $\Delta u - u_t = 0$, $x, y, z \in R$, $t \geqq 0$, $u = 0$ on B, where B is the boundary of the three-dimensional region R, then if one assumes that Green's transformations are permissible, $\int u^{2n} dV$ is a decreasing function of t. In a well known manner, it follows that $\max_R |u|$ is a decreasing function of t. Various consequences of this result are drawn concerning the bounds on particular solutions, and the stability of general solutions. (Received February 5, 1947.)

191. Richard Bellman: *On the stability of solutions of differential-difference equations.* Preliminary report.

It is shown that results concerning stability of solutions can be obtained for differential-difference equations of the form $x'(t) = ax(t) + bx(t+1) + f(x(t), x(t+1))$, where f is a nonlinear function, similar to those obtained for nonlinear differential equations. In particular, if e^{at}, $R(a) < 0$, is a particular solution of the equation with $f \equiv 0$, then there exists a solution of the above equation with prescribed $x(0)$, provided that $|x(0)|$ is sufficiently small, and f is a power series in $x(t)$ and $x(t+1)$ lacking constant and first degree terms. (Received February 5, 1947.)

192. Stefan Bergman and Menahem Schiffer: *On Green's and Neumann's functions in the theory of partial differential equations.*

Let B be a finite plane domain with smooth boundary C and let $P(x, y)$ be continuous and positive in $B+C$. Define the scalar product $D\{\phi, \psi\} = \iint_B [(\partial\phi/\partial x)(\partial\psi/\partial x) + (\partial\phi/\partial y)(\partial\psi/\partial y) + P\phi\psi] dxdy$ of two functions ϕ, ψ in B. A function f will belong to class Ω if it has continuous derivatives in B and satisfies $D\{f, f\} < \infty$. It will belong to class Ω_0 if it belongs to Ω and vanishes on C. It will belong to class Γ if it belongs to Ω and satisfies the differential equation L: $\Delta\phi = P\phi$. A system of functions $\{f_\nu\}$ is called orthonormal if $D\{f_\nu, f_\mu\} = \delta_{\nu\mu}$; it is called complete with respect to a class if every element of this class may be developed into a series of the f_ν. The actual construction of Neumann's function of L in terms of a complete orthonormal system in Ω and Green's function of Laplace's equation is given. An analogous formula for Green's function of L is given (with Ω replaced by Ω_0). The difference between Neumann's and Green's function is the kernal function of a complete orthonormal system in Γ. The formulas contain as a particular case the representation of Green's function in terms of the eigenfunctions of $\Delta\phi - P\phi + \lambda\phi = 0$ with respect to B. (Received March 12, 1947.)

193. R. C. Buck: *Integral valued entire functions.*

Further characterizations of the class of entire functions of exponential type which take integral values at the positive integers are obtained in terms of the growth function $h(\theta, f)$, extending results of Pólya, Selberg, and Pisot. In particular, there are integral valued entire functions of relatively small growth which are not expressible as linear combinations of powers of algebraic integers. (Received March 22, 1946.)

194. R. P. Cesco: *On strong summability.*

Let $A \equiv (a_{n\nu})$ be a regular matrix of real and non-negative numbers $a_{n\nu}$ $(n, \nu = 0, 1, 2, \cdots)$ and consequently strongly regular in the Hamilton-Hill sense (Amer. J. Math. vol. 60 (1938) p. 588). For such matrices the strong limit $[A, p] - \lim s_\nu = s$ of a series $\sum u_\nu$ with partial sums s_ν can be defined uniquely and has the property of finite additivity. These results are obtained: (A) *If* $[A, p] - \lim s_\nu = s$ *and* $s_\nu = O(1)$, *and if* $[A, p] - \lim t_\nu = t$, *then* $[A, p] - \lim s_\nu t_\nu = st$. (B) *If* $[A, p] - \lim s_\nu = s$ *and if* $q \leqq p$, *then* $[A, q] - \lim s_\nu = s$. (C) *If* $[A, p] - \lim s_\nu = s$ *and if* $p \geqq 1$, *then* $A - \lim s_\nu = s$. A matrix A is said to be *strongly permanent* if the series $\sum u_\nu$ strongly summable A remain strongly summable with same order and sum whenever a finite number of zeros is prefixed. It is proved: (D) *If the matrix A is strongly permanent, then the condition* $\lim\inf |u_\nu| = 0$ *is necessary for* $[A, p] - \lim s_\nu = s$. A matrix A is said to be stronger than convergence (or $A \succ E$) if at least one divergent sequence exists which is strongly summable A. It is proved: (E) *In order that $A \succ E$ it is necessary and sufficient that an increasing sequence* $\{\nu_k\}$ *of positive integers exist for which* $\lim \sum_{k=0}^{\infty} a_{n\nu_k} = 0$. (Received March 18, 1947.)

195. Komaravolu Chandrasekharan and Subbaramiah Minakshisundaram: *Some results on double Fourier series.*

Using Bochner's method of spherical summation of multiple Fourier series (Trans. Amer. Math. Soc. vol. 40 (1936) pp. 175–207) the two-dimensional analogues of (i) the Hardy-Littlewood convergence test, (ii) Rogosinski's theorem, and (iii) Bernstein's theorem on the approximation of a function belonging to Lipschitz class by the Cesàro means of its Fourier series (Zygmund, *Trigonometrical series*, pp. 34, 62, 181) are proved, in addition to other results on convergence and summability. (Received March 21, 1947.)

196. Komaravolu Chandrasekharan and Otto Szász: *On Bessel summation.*

$J_\mu(t)$ denotes the Bessel function of order μ: let $\alpha_\mu(t) = 2^\mu \Gamma(\mu+1) t^{-\mu} J_\mu(t)$, so that $\alpha_\mu(0) = 1$. The sequence $\alpha_\mu(nt)$, $n = 0, 1, 2, \cdots$, taken as convergence factors, defines a summability method, considered first by S. Minakshisundaram and K. Chandrasekharan. Thus a series $\sum a_n$ is said to be summable J_μ to the sum s if the series $\sum_{n=0}^{\infty} a_n \alpha_\mu(nt)$ converges in some interval $0 < t < t_0$ to a function $\phi_\mu(t)$, and if $\phi_\mu(t) \to s$ as $t \to 0$. The special case $\mu = 1/2$ gives the well known Lebesgue summability. The authors give some new results concerning the scale of J_μ summability, in particular in comparison to the scale of Cesàro summability, and including known results on Lebesgue summability. The method employs some partially new properties of Bessel functions. (Received March 22, 1947.)

197. Yael N. Dowker: *Invariant measure and the ergodic theorems.*

The ergodic theorems of G. D. Birkhoff and J. von Neumann presuppose an invariant measure. However, one can easily see that if the given measure is potentially invariant, that is, if there exists an invariant measure with no more null sets than the given measure, the ergodic theorems still hold for a broad class of functions (including all bounded measurable functions). It is shown that conversely the ergodic theorems imply that the measure is potentially invariant. Specifically, let S be an abstract space, let m be a measure defined for a Borel family F of sets of S with $m(S) < \infty$ and let T be a 1-1 measurable point transformation of S into itself such that $m(T^{-1}A) = 0$ whenever $m(A)=0$. Then m is potentially invariant if any one of the following conditions hold: 1 (Birkhoff). For every $f(x)\in L(S)$, the averages $F_n(x)(=1/n\sum_{i=0}^{n-1}f(T^ix))$ converge for almost all x. 2 (von Neumann). For every $f(x)\in L^2(S)$, $f(Tx)\in L^2(S)$ and the averages F_n converge in $L^2(S)$. 3. For every descending sequence of sets $A_r\in F$ whose intersection is empty, lim $\sup_{n\to\infty}1/n\sum_{i=0}^{n-1}m(T^{-i}A_r)$ converges to zero. (Received February 17, 1947.)

198. W. F. Eberlein: *A nonlinear differential equation.*

The nonlinear differential equation $u''+8\beta\cos z\sin u=0$, which arises in a vibration problem, is related to both the pendulum equation and the Mathieu equation. Solutions of the type $u=z+a+v$, where $v(z+2\pi)=v(z)$ and $v=O(\beta)$ for small β, are discussed. It is shown that essentially only two exist, one $(a=0)$ being stable, and the other $(a=\pi)$ being unstable. (Received March 21, 1947.)

199. W. F. Eberlein: *A note on ergodic theory.*

Some applications to ergodic theory of a recent theorem of the author on weak compactness (cf. Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947)) are discussed. For example, what appears to be the most general mean ergodic theorem—in the sense of G. Birkhoff and Alaoglu—attainable for Abelian semi-groups of transformations becomes an immediate consequence of a fixed point theorem of A. Markov, an almost elementary proof of which has been sketched by Kakutani (Proc. Imp. Acad. Tokyo vol. 14 (1938)). (Received March 21, 1947.)

200. Bernard Epstein: *Some inequalities relating to conformal mapping upon canonical slit-domains.*

Let a domain D of the complex z-plane containing the point at infinity and bounded by any finite number of smooth curves be mapped conformally and biuniformly upon a domain D_θ of the ζ-plane bounded by rectilinear slits each of which makes the angle θ with the positive direction of the real axis, the mapping function having in the neighborhood of $z=\infty$ a Laurent expansion of the form $\zeta=z+(a_\theta/z)+\cdots$. (The coefficient of each inverse power of z will depend, of course, upon the angle θ.) By setting up a certain positive-definite integral taken over the domain D, and transforming it into an integral taken over the boundary curves, the following inequality is obtained: $\mathrm{Re}(a_\theta e^{-2i\theta})\geqq A/2\pi$, where A is the total area enclosed by the boundary curves. By employing certain results in the theory of systems of orthonormal functions, as developed by S. Bergman and M. Schiffer, it is then shown that the above inequality can be replaced by the following stronger one: $\mathrm{Re}(a_\theta e^{-2i\theta})-|a_\theta|^2/(a_0-a_{\pi/2})\geqq A/2\pi$. It is shown that each of these inequalities is the best possible in the sense that the factor $1/2\pi$ on the right-hand sides cannot be replaced by any larger constant. (Received March 5, 1947.)

201. Paul Erdös and George Piranian: *Over-convergence on the circle of convergence.*

Let the function $f(z)=\sum a_nz^n$ be regular in the unit circle and on the (closed) arc C of the unit circle; let $\phi(x)$ be a function such that $\phi(n)\geqq\log|a_n|$ $(n=0, 1, \cdots)$ and $\phi'(x)\searrow 0$ as $x\to\infty$; and let m_i and n_i be two monotonic sequences such that $a_n=0$ when $m_i\leqq n\leqq n_i$ $(i=0, 1, \cdots)$. If the differences n_i-m_i are sufficiently large in relation to the quantities $\phi(n_i)$, the sequence $s_{m_i}(z)=\sum_{n=0}^{m_i}a_nz^n$ converges to $f(z)$ uniformly on C.

The simplest of the particular results in the paper is the following: If $|a_n|<1$ $(n=0, 1, \cdots)$, $\lim (n_i-m_i)=\infty$, and $k<1$, there exists an integer i_0 such that $|s_{m_i}-f(z)|<\exp\{-(n_i-m_i)^k\}$ on C when $i>i_0$. (Received February 15, 1947.)

202. G. M. Ewing: *Variation problems formulated in terms of the Weierstrass integral.*

An integral of the form $\lim \sum f(x, y; \Delta x, \Delta y)$, introduced by Weierstrass, allows the admission of general rectifiable curves. The present paper is largely expository, bringing together an introduction to existence theorems of the Tonelli type and to related questions for the single-integral parametric problem in ordinary n-space. Results overlap with those of M. Aronszajn, C. Pauc, and L. Tonelli to whose papers the writer is indebted for his interest in the topic. Among the novel features of the present paper is the use of the W-integral in obtaining a simplified proof of upper-reducibility. (Received February 21, 1947.)

203. F. G. Gravalos: *A note on dynamical systems with two degrees of freedom.*

Painlevé stated without proof (*Leçons sur la théorie analytic des équations différentielles*, Paris, 1897, p. 543) that dynamical systems do not admit of integrals of the form $\phi(x_1, \cdots, x_n)=\text{const.}$ where the x's are the coordinates. Using this property, the proof of which is immediate, a rather descriptive theorem is obtained for the case of two degrees of freedom: If a dynamical system with two degrees of freedom admits of a fundamental set of conservative integrals (this terminology is taken from A. Wintner), one at least may be rewritten as a function of the angular momentum $xy'-yx'$, of x' and y', and containing x and y in no other form. (Received March 5, 1947.)

204. Leonard Greenstone: *Mapping by analytic functions.* Part II. *Pseudo-conformal distortion theorems.* Preliminary report.

The author extends results obtained in Part I. *Conformal mapping of multiply connected domains*, Trans. Amer. Math. Soc., to the case of mappings by analytic functions of several variables. In particular the following results are obtained: Let B^4 be a four-dimensional nondegenerate bounded domain, and suppose that b^3, its three-dimensional boundary, contains at least one limit point of third order. (Bergman, *Über die Kernfunktion eines Bereiches und ihr Verhalten am Rand*, I, J. Reine Angew. Math. vol. 169 (1933) pp. 1–42.) If B^4 maps into B_1^4 under pseudo-conformal transformation, then there exists a non-null subdomain $C^4 \subset B^4$ whose image under the transformation is C_1^4, and such that the Euclidean volume of C_1^4 is less than c, where c depends only on B^4 and C^4. An inequality is also given by means of which the distortion of Euclidean length of a Jordan curve under pseudo-conformal mapping may be computed. (Received March 24, 1947.)

205. William Gustin: *A bilinear integral identity for harmonic functions.*

It is shown that any two functions harmonic in open subsets of a euclidean space satisfy a certain bilinear integral identity. The associated quadratic integral identity is used to give a new proof of the theorem that a function harmonic in a connected open set D and vanishing over some non-null open subset of D must vanish throughout D. (Received March 24, 1947.)

206. H. J. Hamilton: *Mertens' theorem and sequence transformations.*

Mertens' theorem on the Cauchy-product for two convergent series is rephrased so as to admit of a valid converse. The resulting theorem is extended to multiple series in two forms. One of these is a dual of a recent theorem of I. M. Sheffer (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1036–1041). The type of convergence used is Pringsheim's and the proofs are effected by means of linear sequence transformation theory. (Received February 21, 1947.)

207. R. G. Helsel and Tibor Rado: *On the Cauchy area of a Fréchet surface.*

Rado (Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 102–106) has stated that $L(S) < \infty$ implies $C(S) = L(S)$, where $L(S)$ is the Lebesgue area of the Frechet surface S and $C(S)$ is a lower semi-continuous functional of S which is based on a construction given by Cauchy. The paper contains a proof of this assertion and makes use of recent results of L. Cesari to show that $C(S) = L(S)$ always. (Received March 15, 1947.)

208. M. R. Hestenes: *Sufficient conditions for isoperimetric multiple integral problem in the calculus of variations.*

In the present paper sufficient conditions are established for the problem of minimizing an integral $I(y) = \int_{S'} f(x_1, \cdots, x_m, y, \partial y/\partial x_1, \cdots, \partial y/\partial x_n) dx_1 \cdots dx_m$ in the class of functions $y(x)$ (x on S) having the same boundary values and satisfying a set of isoperimetric conditions $I_k(y) = \text{constant}$ $(k = 1, \cdots, p)$, where $I_k(y)$ is of the same form as $I(y)$. The sufficient conditions are the direct analogies of the corresponding conditions for the simple integral case and have the same generality (see Hestenes, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 93–118). Conditions for strong and weak relative minima are established. The method used is an extension of the indirect method first introduced by McShane (Trans. Amer. Math. Soc. vol. 52 (1942)) for the problem of Bolza. (Received March 20, 1947.)

209. Rufus Isaacs: *Inverse iterates.*

Let g be a function mapping an arbitrary space E into itself. An inverse iterate of g is a function f of the same type such that always $f(f(x)) = g(x)$. The author has found n.a.s. conditions for the existence of f. By a *linkage* (for g) is meant a subset of E minimally closed under the operation g and its complete inverse. Each linkage for f is the union of two (possibly identical) linkages for g. The existence problem is solved by giving criteria for two distinct linkages for g to be "matable" in this way and for a single linkage to be "self-mating." (Received March 15, 1947.)

210. Mark Kac, Raphael Salem, and Antoni Zygmund: *A gap theorem.*

The main result of the paper is as follows. Let $f(x)$ have period 2π and belong to L^2 and let its mean value be zero. Suppose, in addition, that $\int_0^{2\pi}(f - s_n)^2 dx = O(\log n)^{-\sigma}$, $\sigma > 0$, where s_n denotes the nth partial sum of the Fourier series of f. Let $\{\lambda_k\}$ be any positive sequence such that $\lambda_{k+1}/\lambda_k \geqq q > 1$. Then (1) if $\sigma > 2$, the series $\sum c_k f(\lambda_k x)$ converges almost everywhere provided $\sum c_k^2 \log^2 k < \infty$; (2) if $\sigma \leqq 2$, the series $\sum f(\lambda_k x)/k^{1-\delta}$ converges almost everywhere for every $\delta < \sigma/4$. The relation $1/(m+1)[f(x) + f(\lambda_1 x) + \cdots + f(\lambda_m x)] \to 0$ almost everywhere is a weak consequence of both results. Other theorems of the same type are proved by using properties of quasi-orthogonal functions. (Received February 12, 1947.)

211. L. H. Kanter: *On the roots of orthogonal polynomials and the related Christoffel numbers.*

Under appropriate conditions on the weight function $w(x, \tau)$ the Gauss-Jacobi quadrature formula may be differentiated with respect to the parameter τ, thus giving (1): $\int_a^b \rho(x) w_\tau(x, \tau)dx = \sum_{\nu=1}^{n} \rho(x_\nu)x_\nu'(\tau)\lambda_\nu(\tau) + \sum_{\nu=1}^{n} \rho(x_\nu)\lambda_\nu'(\tau)$ where $\rho(x, \tau)$ is a polynomial of degree $2n-1$ (Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23). If $\rho(x, \tau)$ is $h_\nu(x, \tau)$, the Hermite polynomial of the "first kind," the right side of (1) becomes $\lambda_\nu'(\tau)$. If $w_\tau(x, \tau)$ is positive, then $\lambda_\nu(\tau)$ is an increasing or decreasing function of τ according as $h_\nu(x, \tau)$ is positive or negative in the integration interval. If $\rho(x, \tau)$ is $[l_\nu(x, \tau)]^2[1-h(x-x_\nu)]$, h a parameter, the above results become a special case of $\int_a^b \rho(x) w_\tau(x, \tau)dx = \lambda_\nu(\tau)[h + L''(x_\nu)/L'(x_\nu)]x_\nu'(\tau) + \lambda_\nu'(\tau)$ for $h = -L''(x_\nu)/L'(x_\nu)$. Here $l_\nu(x, \tau)$ is the fundamental polynomial of the Lagrange interpolation (Szegö, *Orthogonal polynomials*). If $h \to +\infty$, then A. Markoff's theorem on the variation of the roots of the orthogonal polynomials with respect to a parameter is obtained. (Received March 21, 1947:)

212. Wilfred Kaplan: *The level curves of a harmonic function.*

The following theorem is established. If F is a regular curve-family filling the xy-plane, then there is a homeomorphism of the xy-plane either onto itself or onto the interior of the unit circle such that F is transformed onto the family of level curves of a harmonic function $u(x, y)$. A regular curve-family is here defined as one locally homeomorphic to a family of parallel lines. (Received January 29, 1947).

213. W. G. Leavitt: *A normal form for matrices whose elements are holomorphic functions.*

This paper considers transformations of type $T^{-1}AT$ of a matrix $A(z)$ whose elements are functions of a complex variable. The results obtained apply to a bounded region R in which all elements of $A(z)$ and all roots of its characteristic equation are holomorphic. The principal theorem states that there exists a matrix $T(z)$ nonsingular throughout R, transforming $A(z)$ into a normal form $U(z)$ whose elements are holomorphic over R and in which all elements below the main diagonal are identically zero. In the course of the proof it is shown that the Weyr characteristic and its associated Jordan form $J(z)$ are definable. A number of results are then established for matrices and vectors of holomorphic functions, leading to the proof that there exists a matrix $S(z)$ whose determinant is not identically zero over R satisfying the equation $AS=SJ$. Finally, $S(z)$ is used in the construction of a matrix $T(z)$ whose determinant is nonvanishing throughout R, and which transforms $A(z)$ into a normal form of type U. (Received March 17, 1947.)

214. D. H. Lehmer: *Approximations to the surface area and electrostatic capacity of the ellipsoid.*

Let a, b, c be the semi-axes of an ellipsoid. Consider all approximations to the surface and capacity of the ellipsoid by means of Minkowsky averages of the form $A\{(1/6)\sum a^\lambda b^\mu c^\nu\}^p$ where the sum extends over all six permutations of (a, b, c). For surface approximation $p=2(\lambda+\mu+\nu)^{-1}$ and $A=4\pi$; for capacity $p=(\lambda+\mu+\nu)^{-1}$ and $A=1$. In this paper, by the proper choice of λ, μ, ν, are found the best possible approximations of this type in the sense that the relative errors, when expanded in a double power series in the essential eccentricities of the ellipsoid, involve terms of the highest possible order. For both problems this order is 10. The λ, μ, ν are, in each case, real algebraic numbers of degree three. (Received March 21, 1947.)

215. D. C. Lewis: *Generalized orthonormality of polynomials.*

Let $a_0(x)$, $a_1(x)$, $\cdots$, $a_p(x)$ be $p+1$ monotonic nondecreasing functions defined for $a \leqq x \leqq b$. Corresponding to an arbitrary sufficiently regular function $f(x)$, introduce the polynomial $P_n(x)$ of degree not greater than n which renders $\sum_{k=0}^{p}\int_a^b(f^{(k)}(x) - P_n^{(k)}(x))^2 da_k(x)$ a minimum. In general $P_n(x)$ is unique, and (as the writer has previously shown) $f(x)-P_n(x)=(1/m!)\int_a^b L_n^m(x, t)df^{(m)}(t)$, where $m \geqq p \geqq 0$ and L_n^m is a certain function independent of f. In the present paper it is assumed that $a_p(x)$ has a nonvanishing derivative. The uniform tendency of $L_n^m(x, t)$ to zero as $n \rightarrow \infty$ (p and m are fixed) is established under certain general conditions. Other asymptotic properties of $L_n^m(x, t)$ are also studied. Methods of proof, adapted from the well known work of Dunham Jackson, are based on Bernstein's bound for the derivatives of a polynomial. Also introduced is a general theory of sets of polynomials $Q_0(x)$, $Q_1(x)$, $Q_2(x)$, $\cdots$ (where $Q_n(x)$ is of degree n), such that $\sum_{k=0}^{p}\int_a^b Q_i^{(k)}(x)Q_j^{(k)}(x)da_k(x)=\delta_{ij}$, where $\delta_{ij}=0$ if $i \neq j$ and $\delta_{ii}=1$. (Received March 20, 1947.)

216. Charles Loewner: *A topological characterization of a class of integral operators.*

A closed oriented curve γ in the x-y-plane will be called of non-negative circulation if its order relative to any point not on γ is non-negative. Let $k(t)$ be an L-integrable function of t with the period 2π. Consider the integral operator $y(t)=\int_0^{2\pi}k(\tau)x(t-\tau)d\tau$ applied to continuous functions $x(t)$ having the same period 2π. $y(t)$ is then also continuous of period 2π. The functions $x=x(t)$, $y=y(t)$ give a parametric representation of a closed oriented curve in the x-y-plane. Any curve obtained in this way will be called "generated by the kernel $k(t)$." Various problems of analysis and geometry lead to the question: Which kernels generate *only* curves of non-negative circulation? A complete answer is given in this paper: The class of kernels in question consists of those L-integrable functions $k(t)$ analytic in the open interval $0<t<2\pi$ whose derivative $k'(t)$ can be represented by a Laplace-Stieltjes integral $k'(t)=\int e^{-rt}d\alpha(r)$ $(-\infty<r<\infty)$ with a non-negative mass distribution $d\alpha(r)\geqq 0$. (Received March 12, 1947.)

217. E. R. Lorch: *On certain implications which characterize Hilbert space.*

Two principal characterizations of Hilbert space are given: (1) Let $\mathfrak{B}$ be a vector space with the property: There exists a real number α ($\alpha \neq 0, 1$) such that if a vector pair f, g satisfies $|f+g|=|f-g|$ it follows that $|f+\alpha g|=|f-\alpha g|$. Then $\mathfrak{B}$ is a Hilbert space. (2) Let $\mathfrak{B}$ be a vector space with the property: If a vector pair f, g satisfies $|f|=|g|$ there follows that $|\alpha f+\alpha^{-1}g| \geqq |f+g|$ where α is an arbitrary nonzero real number. Then $\mathfrak{B}$ is a Hilbert space. Five new characterizations of Hilbert space are given each of which rests on Theorem 1 above. Among them are the following extensions of a theorem of Jordan and von Neumann: (3) If in $\mathfrak{B}$ there exists for arbitrary f, g a (nontrivial) relation involving the quantities $|f|$, $|g|$, $|f+g|$, and $|f-g|$, then $\mathfrak{B}$ is a Hilbert space. Note that the hypothesis merely calls for the existence of a relation. (4) If for arbitrary vectors f, g, h in $\mathfrak{B}$ with $f+g+h=0$ it is true that $3\{|f|^2+|g|^2+|h|^2\}=|f-g|^2+|g-h|^2+|h-f|^2$, then $\mathfrak{B}$ is a Hilbert space. (Received February 17, 1947.)

218. C. W. Mathews: *Cauchy type double integral representations for functions of a complex variable.*

Consider the class, R, of functions of the complex variable which are continuous in a bounded domain, G, such that (1) $\lim\sup (1/|I|)\int_{(I)}f(z)dz<\infty$ (for intervals containing z_0) for all z_0 in $G-S$ where S is a denumerable infinity (at most) of segments parallel to the axes and (2) such that the partial derivatives of the real and imaginary parts of the function with respect to the real and imaginary parts of the variable exist almost everywhere in G. In a paper by Trjitzinsky (to appear in Journal de Mathématiques) conditions were found under which the function, $f(z)$, could be represented as a Cauchy type double integral. The author has determined conditions under which this integral equation is equivalent to a regular Fredholm integral equation. Hence, if $f(z)$ belongs to R and $K(z, \zeta)$ satisfies certain conditions while the regular Fredholm integral equation, $2\pi i\Phi(z)-\iint_G[\int_{(I)}K(z, \zeta)\Phi(\zeta)d\zeta_1 d\zeta_2=-\eta'(z)$, where $\eta(z)=\int_{(I)}f(z)dz$ for I in G, has a solution, Φ, then $f(z)$ has the representation $f(z)=-\iint_G k(z, \zeta)\Phi(\zeta)/(\zeta - z)d\zeta_1 d\zeta_2 + a(z)$ where $a(z)$ is analytic, $\zeta = \zeta_1 + i\zeta_2$, and $k(z, \zeta)/(\zeta-z)=1/(\zeta-z)+K(z, \zeta)$. (Received February 21, 1947.)

219. E. J. Mickle: *Metric foundations of continuous transformations.*

Let T be a continuous transformation from an analytic set A in a complete and separable metric space M into a metric space M^*, in which finite Carathéodory outer measures ν and ν^* are defined respectively. Let $N(p^*, E)$ designate the number of inverse points a point p^* of M^* has in a subset E of A. T is called of bounded variation if $N(p^*, E)$ is a ν^*-summable function and, if of bounded variation, is called absolutely continuous if sets of ν-measure zero are taken into sets of ν^*-measure zero. If T is of bounded variation, a completely additive class of subsets E of M containing closed sets can be found on which $\mu(E)$, the ν^*-integral of $N(p^*, E)$ over M^*, is a completely additive set function. The absolutely continuous part of μ can be expressed as the ν-integral of a point function which plays the role of the Jacobian of the transformation. Formulas (involving this Jacobian) for transforming definite integrals are obtained in this paper. (Received March 9, 1947.)

220. C. N. Moore: *Generalized limits in general analysis.* III.

In volumes 24 and 25 of Trans. Amer. Math. Soc. there were published two papers, entitled respectively: *Generalized limits in general analysis, First paper* and *Generalized limits in general analysis, Second paper*. It was the aim of these papers to develop a general theory in the field of general analysis, which would include as special cases results concerning summable series and their analogues for integrals. The applications of such a theory were illustrated by proving a general theorem concerning the equivalence of C and H summability and the analogous result for integrals. In the present paper a further application is made by proving a theorem which includes the regularity theorem for summability methods of a certain type, due to Silverman and Toeplitz, and the analogous theorem for integrals, due to Agnew. (Received March 8, 1947.)

221. John von Neumann and E. R. Lorch: *On the euclidean character of the perpendicularity relation.*

A proof is given of the following theorem: Let $\mathfrak{B}$ be a vector space such that whenever two vectors f and g satisfy $|f|=|g|$, it follows that $|\alpha f+\beta g|=|\beta f+\alpha g|$ for all real α and β. Then $\mathfrak{B}$ is a Hilbert space (of unspecified dimensionality). The theorem has been proved by F. A. Ficken (Ann. of Math. (2) vol. 45 (1944) pp. 362–366). The present proof which is very short introduces a notion of perpendicularity in vector spaces which renders transparent the geometric phenomena in which the problem is rooted. This same notion has application to the further study of the characterization of Hilbert space by means of implications which involve its norm. (Received February 17, 1947.)

222. T. G. Ostrom: *The solution of linear integral equations by means of Wiener integrals.*

Let the Fredholm solution of the integral equation $z(t)=x(t)+\int_0^1 k(t, s)x(s)ds$ be given by $x(t)=z(t)+\int_0^1 R(t, s)z(s)ds \equiv G[z|t]$. By applying two linear transformations to the Wiener integral $\int_c^w G[z+y|t]d_w y$, the author shows that under fairly general conditions the solution can be expressed as a Wiener integral whose integrand is a functional involving only the fixed functions K and z, the variable function of integration, and the number t. The Fredholm determinant D (which is also expressible as a Wiener integral) appears as a constant factor and hence can be obtained without further computation by substitution in the integral equation. (Received March 15, 1947.)

223. Tibor Rado and E. J. Mickle: *A new geometrical interpretation of the Lebesgue area of a surface.*

Let S refer to a Fréchet surface of the type of the 2-cell in Euclidean 3-space E_3, let G_4 be the 4-dimensional space of all lines $g \in E_3$, and let m denote the properly normalized invariant measure in G_4. For each line $g \in E_3$ an essential intersection number $\kappa(g)$ is introduced in terms of ϵ-deformations of S. In this paper it is shown that $\int \kappa(g)dm$ taken over G_4 is equal to the Lebesgue area of S. (Received March 19, 1947.)

224. O. W. Rechard and P. V. Reichelderfer: *A new criterion for the extension of rectangle functions.*

Let $\phi(R)$ be a real, finite, non-negative function defined on the class K of rectangles R contained in a fixed oriented closed rectangle F. It is shown that ϕ admits a completely additive extension to an additive class of sets including all Borel sets if and only if it satisfies the following conditions: (i) if R_1 and R_2 are mutually exclusive rectangles in K contained in a rectangle R in K, then $\phi(R_1)+\phi(R_2) \leqq \phi(R)$; (ii) if R is a rectangle in K covered by two rectangles R_1 and R_2 in K, then $\phi(R) \leqq \phi(R_1) + \phi(R_2)$; (iii) if R is any rectangle in K and R_ϵ is the common part of F and the rectangle containing R with sides parallel to the corresponding sides of R at distance ϵ, then $\lim \phi(R_\epsilon)=\phi(R)$ as ϵ tends to zero through positive values. A similar result is established for functions of oriented open rectangles. In this case, R_ϵ is the rectangle contained in R with sides parallel to the corresponding sides of R at distance ϵ. Functions defined on all closed rectangles or on all open rectangles are also considered. (Received March 10, 1947.)

225. P. V. Reichelderfer: *The effect of a lipschitzian transformation on the area of a continuous surface.*

A transformation $X(x)$ from euclidean three-space into euclidean three-space is termed lipschitzian M if M is a positive constant such that the distance between the images $X(x_1)$ and $X(x_2)$ of any pair of points x_1 and x_2 does not exceed M times the distance between x_1 and x_2. Any continuous surface S is transformed by $X(x)$ into a continuous surface $X(S)$. If $X(x)$ is lipschitzian M, it is shown that the lebesgue area of $X(S)$ cannot exceed the square of M times the lebesgue area of S. Various applications are considered. (Received March 15, 1947.)

226. E. H. Rothe: *Extrema of functions in Banach spaces.* Preliminary report.

Let V be a bounded domain in a Banach space E. Let $I(x)$ be a real-valued function defined and (strongly) continuous in V. Moreover it is assumed that the first and second differential of $I(x)$ exist and are completely continuous and that the latter is "Hessian," that is, satisfies a certain condition which corresponds to the non-vanishing of the Hessian determinant in the finite-dimensional case. It is proved that then $I(x)$ is also weakly continuous. As a consequence $I(x)$ reaches a maximum and a minimum value in any weakly compact subset of V, and the existence of a maximum and minimum of $I(x)$ in any closed bounded set can be asserted for a number of Banach spaces important for applications to analysis, for example, for reflexive spaces (by the use of a theorem by Alaoglu, Ann. of Math. vol. 41 (1940)) or the Schauder spaces (Math. Ann. vol. 106 (1932) p. 663). (Received March 21, 1947.)

227. A. C. Schaeffer and D. C. Spencer: *A general class of problems in conformal mapping.*

Let S be the class of functions $f(z)$ which are regular and schlicht in the unit circle $|z|<1$ and are normalized by the condition that $f(0)=0, f'(0)=1$. Let R be a closed set in $|z|<1$, and let $\psi_\nu(\alpha)$ be a measure function defined in the space R. If F_ν is a function of $f\subset S$ and its derivatives up to the nth order and $P_\nu=\int F_\nu d\psi_\nu$ then (P_1, P_2, P_m) is a point in a euclidean space of m dimensions. The problem is to find the region of variability of this point. Several unsolved classical problems are special cases of this general problem. The region of variability of $f'(z_1)$ where $|z_1|<1$ is one example, and a differential equation is obtained which defines this region. The differential equation can be integrated explicitly in terms of elementary functions. (Received March 22, 1947.)

228. H. M. Schaerf: *Properties of measures biinvariant under a composition law.* Preliminary report.

Let the measure space R be the union of a sequence of measurable sets of finite measure. Let a composition law ascribe to every $x, y\in R$ an element $xy\in R$ so that Weil's condition M is satisfied and so that the measure of any measurable set X is invariant under the transformations rX, Xr^{-1} for every $r\in R$. The author gives a simple proof of the uniqueness of such a measure. He proves, moreover, that any set A, whose measure is not greater than that of a set B, can be split into a sequence of measurable sets, whose images under the composition law are disjoint subsets of B, and into a set of measure zero. Therefore, any value of $m(X)$ less than $m(B)$ is assumed on some subset of B. Furthermore, any one of two sets with equal measures can be split into a sequence of disjoint subsets and a set of measure zero so that subsets with equal subscripts are images of each other under the composition law. (Received March 20, 1947.)

229. H. M. Schaerf: *Two theorems on measure spaces with a composition law.* Preliminary report.

Let the measure space R be the union of a sequence of measurable sets of finite measure. Let a composition law ascribe to every x, $y \in R$ an element $xy \in R$ so that Weil's condition M is satisfied and so that, for every set X of measure zero and every element r of R, the set rX is of measure zero. By means of the generalized theorem of Fubini the following theorems are proved: (1) For any two sets A, B with positive measures there is an element r such that Ar^{-1} meets B in a set of positive measure. (2) If Z is a set of measure zero, then any completely additive function of a measurable set vanishes on Zr^{-1} for almost all $r \in R$ (generalization of a theorem of Wiener-Young). If to every element $x \in R$ a transformation t_x of R in R is ascribed, then generalizations of the above theorems are furnished by the composition laws $xy = t_x(y)$, and $xy = t_y(x)$. (Received March 20, 1947.)

230. I. M. Sheffer: *A limit theorem.*

A classical result in trigonometric series theory is that: If (i) $a_n \cos nx + b_n \sin nx \to 0$ for all x on an interval I, then $a_n \to 0$, $b_n \to 0$. In complex form (i) is replaced by (ii) $c_n \exp \{nx\} + d_n \exp \{-nx\} \to 0$. Here $\exp \{u\} \equiv e^{iu}$. The present note extends this result in various directions. For example: Let $\{a_{s,n}\}$, $s = 1, \cdots, k$, be complex number sequences, and let $\{r_{s,n}\}$ be real sequences with the following property: None of the sequences $\{r_{s,n} - r_{p,n}\}$ $(s \neq p)$ has zero as a limit point. If (iii) $\sum_{s=1}^{k} \alpha_{s,n} \exp \{r_{s,n}x\} \to 0$ for all x on I, then $a_{s,n} \to 0$ $(s = 1, \cdots, k)$. (Received March 19, 1947.)

231. Y. C. Shen: *Interpolation to some classes of analytic functions by functions with pre-assigned poles.*

Let S_α be the class of functions $f(z)$ regular for $|z| < 1$ and such that for some $\alpha > 1$ the integral $\int_0^r \int_0^{2\pi} (1-\rho)^{\alpha-2} |f(\rho e^{i\theta})|^2 \rho d\rho d\theta$ is bounded for $r \to 1$. Let a_{nk}, $k = 1, 2, \cdots, n$; $n = 1, 2, \cdots$, be a system of points in $|z| < 1$, without points of accumulation in $|z| < 1$. Let $\Phi_{nk}(z) = (1 - \bar{a}_{nk} z)^{-\alpha}$, and let $f_n(z)$ denote the function of the form $\sum_1^r A_{nk} \Phi_{nk}(z)$ interpolating $f(z)$ at the points a_{nk}, $k = 1, 2, \cdots, n$. Necessary and sufficient conditions are obtained for the sequence a_{nk} so that for every $f \in S_\alpha$ the sequence $\{f_n(z)\}$ converges uniformly to f in every closed circle interior to $|z| < 1$. (Received February 17, 1947.)

232. R. H. Stark: *Some classes of monotone functions.*

The purpose of this paper is the investigation of classes of monotone, continuous functions $h(x)$ made up from a system of basis functions $g_i(x)$ as sums of Hellinger integrals, $h(x) = \sum_{i=1}^{\infty} \int_0^x (df_i)^2 / dg_i$. Classes defined by different basis functions are discussed. Criteria for determining whether a function belongs to a particular class are given. There is presented a method for constructing a set G of monotone continuous functions such that a continuum of classes without common elements is determined by choice of bases from G, and yet there exist functions outside of all classes determined by such a choice. (Received March 19, 1947.)

233. C. F. Stephens: *Concerning linear and nonlinear difference equations.*

The author considers the system of equations $u_i(x+1) = x^k \sum_{j=1}^{n} h_i^{(j)}(x) u_j(x)$

$+x^k g_i(u_1(x), \cdots, u_n(x); x)$ $(i=1, \cdots, n)$, where k is any integer or zero, $h_i^{(j)}(x)$ are bounded and continuous (not necessarily analytic) in the neighborhood of infinity, g_i are polynomials in the $u_i(x)$, but not in x, whose coefficients are continuous functions of x in the neighborhood of infinity. The nth order nonlinear difference equation corresponding to the above system, the nth order homogeneous linear difference equation and the case where $g_i \equiv 0$ are also considered. In all cases the author obtains unique particular solutions which are continuous and bounded in a certain domain extending to infinity on the left. The method used in this paper is based on some previous results of the author concerning the solutions of nonlinear difference equations for large values of the independent variable (Bull. Amer. Math. Soc. vol. 50 (1944) p. 343). (Received February 21, 1947.)

234. Walter Strodt (National Research Fellow): *Note on quasi-harmonic functions.* Preliminary report.

A quasi-harmonic function of x and y, as defined by Garrett Birkhoff, is a function satisfying the Gauss mean-value theorem on all circles of radius unity. This note answers affirmatively the question, raised by Birkhoff, whether there exist non-harmonic functions of x and y which are quasi-harmonic and also analytic in x and y. For example, $\exp(ax+by)$ is such a function if $a^2+b^2 \neq 0$ and $F(a, b) = \int_0^{2\pi}[1-\exp(a\cos\theta+b\sin\theta)]d\theta=0$. For every fixed a there exist infinitely many b such that $F(a, b)=0$, since otherwise $F(a, b)$, which is an even transcendental entire function of b, of order unity, would be a polynomial times an exponential of an entire function, which is absurd. (Received March 17, 1947.)

235. L. V. Toralballa: *The integral in a normed division quasi-ring.*

Consider a system R with two operations, $+$ and $\times$, satisfying: (1) R is an additive Abelian group; (2) R is closed with respect to $\times$; (3) $\times$ is associative; (4) there exists a unity; (5) every nonzero element of R has a unique inverse; (6) to every element a of R corresponds a non-negative real number $|a|$, called its norm, satisfying (a) $|a|=0$ if and only if $a=0$; (b) $|a\cdot b|=|a|\cdot|b|$; (c) $|a|=|-a|$; (d) $|a+b| \leqq |a|+|b|$; (e) there exists a real number $N>0$ such that $|ca-cb| \leqq N|c(a-b)|$ always; (f) if cosine $(p, q) \equiv (|p|^2+|q|^2-|p-q|^2)/2|p||q|$, $p\neq 0$, $q\neq 0$, there exists a real number $M>0$ such that $(ca, cb) \leqq M(a, b)$ always. Consider functions f on R to R. In the norm topology one defines such notions as "curve" and "continuity." A regular curve C is one in which (1) C is rectifiable; (2) the limit of the ratio of the length of chord to that of the subtended arc is unity; (3) the scalar curvature is a continuous function of the arc length. The integral $\int_C f(\xi)d\xi$ is defined to be lim as norm of subdivision$\to 0$ of $\sum(z_\nu - z_{\nu-1})f(\xi_\nu)$. This is proved to exist whenever C is regular and f is continuous over C. (Received March 20, 1947.)

236. W. R. Utz: *On the decomposition of meromorphic functions.*

Let $f(z)$ be defined and meromorphic in a bounded and simply-connected region R having locally connected boundary $F(R)$. Let z_0 be a point of $F(R)$. If for each crosscut c subdividing R into subregions R_1 and R_2, where $z_0 \in F(R_1)-\bar{c}$, there exists a region $R_1' \subset R_1$ determined by a crosscut c', of R, and a rational function $g(\lambda)=w$, such that $z_0 \in F(R_1')-\bar{c}'$ and $g^{-1}f(z)$ is 1-1 and meromorphic on R_1', then $f(z)$ is said to be *rationally-multivalent.* It is shown that a necessary and sufficient condition that $f(z)$ be decomposable into a 1-1, bounded and regular function on R, followed by a rational function, is that it be rationally-multivalent at each point of $F(R)$. This is a generalization of a result of L. H. Loomis (*The decomposition of meromorphic functions into rational functions of univalent functions*, Trans. Amer. Math. Soc. vol. 50 (1941) pp. 1–14). (Received March 7, 1947.)

237. J. L. Walsh: *The location of the critical points of simply and doubly periodic functions.*

By the use of conformal transformations, especially those involving the exponential function, known theorems (for instance that of Lucas) concerning the zeros of the derivative of a rational function and the critical points of a harmonic function yield new results on the zeros of the derivative of a periodic analytic function and the critical points of a periodic harmonic function. (Received March 6, 1947.)

238. Alexander Weinstein: *On Dirichlet's discontinuous factor and a class of many-valued functions.*

Continuing previous investigations (Bull. Amer. Math. Soc. vol. 53 (1947) p. 59) the paper deals with the function $\alpha(x, y) = \pi y^{-q} b^{-q} \int_0^\infty \exp(-xt) J_q(yt) J_{q+1}(bt) dt$ $(x \geqq 0,\ b > 0,\ -1 < q \leqq -1/2)$, which represents a branch of a many-valued function with the ramification points $x = 0$, $y = \pm b$. For $q = -1/2$, $\alpha(x, y) = 2\int_0^\infty t^{-1} \exp(-xt) \cos(yt) \sin(bt) dt$. It is shown that, in this case, α is the angle at (x, y) subtended by the segment $(-b, b)$ of the y-axis. This angle $\alpha(x, y)$ is the streamfunction of doublets distributed along the segment, and $\pi^{-1}\alpha(0, y)$ is Dirichlet's discontinuous factor. (Received February 26, 1947.)

239. D. V. Widder: *Inversion formulas for convolution transforms.*

The author discovered earlier a linear differential operator of infinite order which inverts the Stieltjes transform, $f(x) = \int_0^\infty (x+t)^{-1}\phi(t) dt$. This is a convolution transform, $f(x) = \int_{-\infty}^\infty G(x-t)\phi(t) dt$, with $G(x) = (1+e^{-x})^{-1}$ after an exponential change of variable, and the inversion operator is $\pi^{-1} \sin \pi D$. Here D means differentiation with respect to x and the operator is interpreted by means of the infinite product expansion of the sine function. In the present paper the author generalizes this result, replacing $G(x)$ by any function whose bilateral Laplace transfunction is the reciprocal of an entire function $E(s) = s\prod_1^\infty [1-(s/a_k)^2]$, $\sum_1^\infty a_k^2 < \infty$. The inversion operator becomes $E(D)$. The method of Green's functions for differential systems is used. The kernel $G(x)$ is interpreted as a Green's function for a linear differential system of infinite order. (Received February 4, 1947.)

240. Albert Wilansky: *An application of Banach linear functionals to the theory of summability.* Preliminary report.

Normal conservative summability matrices are classified as co-regular, co-null. In particular multiplicative r ($r \neq 0$, $r = 0$) matrices are, respectively, co-regular, co-null. A co-null summability field cannot be a subfield of a co-regular field. Every field is a proper subfield of a co-null field. Theorems known for regular matrices (Mazur, Steinhaus, Agnew, Hill, and so on) are extended to conservative matrices, for example a co-regular matrix cannot sum all bounded sequences, Mazur's consistency theorem holds for co-regular matrices only. A normal matrix is shown to be of type M if and only if its columns and another sequence "span" (c). Two normal matrices with the same field are both or neither of type M, thus "of type M" may be applied to fields.

If F is called regular if a normal regular matrix exists with field F, a regular field is co-regular; F is regular if and only if $f(x) = \lim x_n$ is continuous on (c) using the "Mazur norm" $\|x\| = \sup |\sum_{k=0}^{n} a_{nk} x_k|$. Subfields of regular fields are regular. Other conditions are given. All Mazur metrics for a given field are equivalent and all fields are separable. A field is a non-reflexive Banach space. A normal conservative matrix sums only convergent sequences if and only if $\|A^{-1}\| < \infty$ where $\|A\| = \sup \sum_{k=0}^{n} |a_{nk}|$. (Received March 20, 1947.)

241. J. E. Wilkins: *Neumann series of Bessel functions.*

Suppose that $|f(t)|$ is integrable over $(0, a)$ and that $t^{-3/2}|f(t)|$ is integrable over (a, ∞) whenever $a > 0$. Let $s = \sum_{n=0}^{\infty} a_{2n+1} J_{2n+1}(x)$, where $a_{2n+1} = \int_0^{\infty} t^{-1} f(t) J_{2n+1}(t) dt$, be the Neumann series associated with $f(x)$. It is shown that $s = \{f(x+) + f(x-)\}/2$ at each positive point x in a neighborhood of which $f(t)$ is of bounded variation if and only if $\int_1^{\infty} r J_0(xr) dr \int_a^N f(t) J_0(tr) dt$ converges to zero as a approaches zero and N approaches ∞. (Received March 21, 1947.)

Applied Mathematics

242. J. W. Calkin: *Incipient shock waves in one dimension.*

The author considers the following one-dimensional model: a rectilinear cylinder, closed at one end by a piston, and containing a perfect gas. As is in effect known, while a discontinuous increase in piston velocity produces an immediate shock wave, starting at the piston face, a continuous acceleration results in a delayed shock which starts at a positive distance from the piston face, with initial velocity that of sound. The present paper is concerned with the boundary value problem to which the study of the subsequent motion leads, and provides procedures for approximating to the solution. A variant of the hypothesis of a perfect gas is also considered. (Received March 21, 1947.)

243. J. B. Díaz and H. J. Greenberg: *The determination of upper and lower bounds for the deflection of a clamped plate.*

A method is given for the determination of upper and lower bounds for the deflection w at any point of a thin elastic plate of arbitrary shape clamped along its edges and subjected to a distributed load. The method is based on the application of two variational principles: the first is that of minimum potential energy, the second is closely related to Castigliano's principle of minimum complementary energy but does not seem to have been used before in the present form. These principles yield inequalities for an integral of w. By considering two auxiliary loadings of the plate, in addition to the given loading, inequalities are obtained directly for w at any specified point. The bounds are obtained in terms of integrals of certain admissible functions. Explicit iterative formulas are given by means of which sequences of admissible functions can be utilized to successively improve the bounds. (Received March 21, 1947.)

244. J. B. Díaz and Alexander Weinstein: *Schwarz' inequality and the methods of Rayleigh-Ritz and Trefftz.*

It is shown that lower and upper bounds of a quadratic functional can be obtained by a simple and direct application of Schwarz' inequality and Green's formula, the results being equivalent to the application of the methods of Trefftz and Rayleigh-