A PARTICULAR GENERALIZED LAPLACIAN

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In a preceding paper, the author developed a generalized Laplacian for functions having subharmonic logarithms [2].[1] The purpose of this note is to indicate how generalized Laplacians may be used to weaken differentiability requirements; in particular, the generalized Laplacian

$$\Delta^* f(x, y) \equiv \limsup_{\rho\to 0} \frac{4}{\rho^2} [L(f; x, y; \rho) - f(x, y)] \tag{1}$$

is used to weaken differentiability requirements in certain theorems due to Kierst and Saks [4] and the author [3].

The definitions and notation used in [2] will be used here. In addition, use will be made of the following known result.

THEOREM A [1]. *If $f(x, y)$ is continuous in a domain G, then a necessary and sufficient condition that $f(x, y)$ be subharmonic in G is that*

$$\Delta^* f(x, y) \geqq 0 \tag{2}$$

hold throughout G.

A slightly more general version of a theorem due to Kierst and Saks is the following one.

THEOREM 1. *Let $F(t)$ have a continuous second derivative, with $F'(t) > 0$, for $-\infty < t < \infty$. If $f(x, y)$ has continuous partial derivatives of the first order in a domain G, and if $F[\alpha x + \beta y + f(x, y)]$ is subharmonic in G for every choice of the real constants α, β, then $f(x, y)$ is subharmonic in G.*

PROOF. Let (x_0, y_0) be a fixed, arbitrary point of G. Then after expanding $F(t)$ and $f(x, y)$ in Taylor series about $t_0 \equiv \alpha x_0 + \beta y_0 + v(x_0, y_0)$ and (x_0, y_0), respectively, one obtains

$$\begin{aligned} L(\phi_{\alpha,\beta}; x_0, y_0; \rho) - \phi_{\alpha,\beta}(x_0, y_0) \\ &= F'(t_0)[L(f; x_0, y_0; \rho) - f(x_0, y_0)] \\ &\quad + \frac{\rho^2 F''(t_0)}{4} [(\alpha + f_x)^2 + (\beta + f_y)^2] + o(\rho^2), \end{aligned} \tag{3}$$

where

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$\phi_{\alpha,\beta}(x, y) \equiv F[\alpha x + \beta y + f(x, y)],$$

$$f_x \equiv \left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)}, \qquad f_y = \left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)},$$

and $o(\rho^2)$ is a quantity (not always the same quantity) such that

$$\lim_{\rho \to 0} \frac{o(\rho^2)}{\rho^2} = 0. \tag{4}$$

Now set $\alpha = -f_x$, $\beta = -f_y$. Since $\phi_{\alpha,\beta}(x, y)$ is subharmonic in G for all choices of the real constants α, β, it follows from (1), (3), (4) and Theorem A that (2) holds at (x_0, y_0). But (x_0, y_0) was an arbitrary point of G, so that (2) holds throughout G; therefore, by Theorem A, $f(x, y)$ is subharmonic in G. This completes the proof.

In a similar manner one may prove the following more general version of a theorem due to the author [3].

THEOREM 2. *Let $F(t)$ and $f(x, y)$ have the properties noted in Theorem 1. If the function $F\{\log[(x-\alpha)^2+(y-\beta)^2]+f(x, y)\}$ is subharmonic in G for every choice of the real constants α, β, then $f(x, y)$ is subharmonic in G.*

The same technique may be applied to other results [1, 3] to obtain slightly more general theorems. However, it would be desirable to remove all conditions of differentiability (on $f(x, y)$)—which the usual averaging process does not appear to do.

Other generalized Laplacians may be used to obtain results similar to those above; for example, either

$$\limsup_{\rho \to 0} \frac{8}{\rho^2} [A(f; x, y; \rho) - f(x, y)],$$

or

$$\limsup_{\rho \to 0} \frac{8}{\rho^2} [L(f; x, y; \rho) - A(f; x, y; \rho)]$$

may be used.

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4. S. Saks, *On subharmonic functions*, Acta Univ. Szeged. vol. 5 (1930–1932) pp. 187–193.

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