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ON THE $(C, 1)$ SUMMABILITY OF CERTAIN RANDOM SEQUENCES

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It is known [1][1] that if a sequence $\{a_n\}$ $(n=1, 2, \cdots)$ of real numbers is summable $(C, 1)$ to a value α, and if $\sum a_n^2/n^2 < \infty$, then almost all the subsequences of $\{a_n\}$ are summable $(C, 1)$ to α. It will be shown that this statement continues to hold if "almost all" is replaced by "with probability 1" and "subsequences" by the more general term "product sequences," the meaning of which will be defined in the next paragraph. The only analytic tool used is the strong law of large numbers [2]: if $\{y_n\}$ is a sequence of independent random variables with expected values $E(y_n)=0$ and $E(y_n^2)=b_n^2$, for which $\sum b_n^2/n^2 < \infty$, then with probability 1 the sequence $\{y_n\}$ is summable $(C, 1)$ to the value 0.

DEFINITION. Let $\{a_n\}$ be a sequence of constants and let $\{x_n\}$ be a sequence of random variables such that the values of each x_n are non-negative integers. For every n let $k(n)$ be the least positive integer m such that

$$(1) \qquad \sum_{1}^{m} x_i \geqq n,$$

Presented to the Society, September 17, 1945; received by the editors January 23, 1946.

[1] Numbers in brackets refer to references listed at end of paper.

and set $w_n = a_{k(n)}$. (In intuitive terms this definition of w_n is equivalent to the following. Starting with the sequence

$$x_1 a_1,\ x_2 a_2,\ \cdots,\ x_n a_n,\ \cdots, \tag{2}$$

strike out each term for which $x_i = 0$, and then replace each remaining term $x_i a_i$ by the block $a_i, a_i, \cdots, a_i$ with x_i terms. Then w_n is the nth term in the resulting sequence.) The sequence $\{w_n\}$ of random variables will be called the product of $\{a_n\}$ by $\{x_n\}$.

THEOREM 1. *Let $\{a_n\}$ be a sequence of real numbers summable $(C, 1)$ to the value α. Let $\{x_n\}$ be a sequence of independent, non-negative, integer-valued random variables with $E(x_n) = \bar{x}$ $(0 < \bar{x} < \infty)$, $E(x_n - \bar{x})^2 = d_n^2$, and such that*

$$\sum_1^\infty \frac{d_n^2}{n^2} < \infty, \tag{3}$$

$$\sum_1^\infty \frac{a_n^2 d_n^2}{n^2} < \infty. \tag{4}$$

Then with probability 1 the product sequence $\{w_n\}$ is summable $(C, 1)$ to the value α.

PROOF. It follows from the strong law of large numbers applied to the sequence $\{x_n - \bar{x}\}$ that with probability 1,

$$\begin{aligned} &\lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \\ &\qquad = \lim_{n\to\infty} \left[\frac{(x_1 - \bar{x}) + \cdots + (x_n - \bar{x})}{n} + \bar{x} \right] = \bar{x} \neq 0. \end{aligned} \tag{5}$$

Likewise it follows from the same law applied to the sequence $\{a_n(x_n - \bar{x})\}$ that with probability 1,

$$\begin{aligned} &\lim_{n\to\infty} \frac{x_1 a_1 + \cdots + x_n a_n}{n} \\ &= \lim_{n\to\infty} \left[\frac{a_1(x_1 - \bar{x}) + \cdots + a_n(x_n - \bar{x})}{n} + \bar{x}\, \frac{a_1 + \cdots + a_n}{n} \right] = \bar{x}\alpha. \end{aligned} \tag{6}$$

But

$$\frac{x_1 a_1 + \cdots + x_n a_n}{x_1 + \cdots + x_n} = \left[\frac{x_1 a_1 + \cdots + x_n a_n}{n} \right] \left[\frac{n}{x_1 + \cdots + x_n} \right]. \tag{7}$$

Hence with probability 1,

$$\lim_{n\to\infty} \frac{x_1a_1 + \cdots + x_na_n}{x_1 + \cdots + x_n} = \alpha. \tag{8}$$

Moreover, from (5) we note that, with probability 1, $\lim (x_1 + \cdots + x_n) = \infty$, and hence, with probability 1, $k(n)$ is defined for every n.

Now we introduce the abbreviations

$$r(n) = n - (x_1 + \cdots + x_{k(n)-1}), \tag{9}$$

$$p(n) = x_1a_1 + \cdots + x_{k(n)-1}a_{k(n)-1}, \tag{10}$$

$$q(n) = x_1 + \cdots + x_{k(n)-1}, \tag{11}$$

noting that from the definitions it follows that

$$0 < r(n) \leqq x_{k(n)}, \tag{12}$$

and that

$$\frac{w_1 + \cdots + w_n}{n} = \frac{a_{k(1)} + \cdots + a_{k(n)}}{n} = \frac{p(n) + r(n)a_{k(n)}}{q(n) + r(n)}. \tag{13}$$

There are now two cases to consider. If $a_{k(n)} \geqq p(n)/q(n)$ then it is easily seen from (13) that

$$\begin{aligned} \frac{x_1a_1 + \cdots + x_{k(n)-1}a_{k(n)-1}}{x_1 + \cdots + x_{k(n)-1}} &= \frac{p(n)}{q(n)} \leqq \frac{w_1 + \cdots + w_n}{n} \\ &\leqq \frac{p(n) + x_{k(n)}a_{k(n)}}{q(n) + x_{k(n)}} = \frac{x_1a_1 + \cdots + x_{k(n)}a_{k(n)}}{x_1 + \cdots + x_{k(n)}}. \end{aligned} \tag{14}$$

However, if $a_{k(n)} \leqq p(n)/q(n)$ then (14) holds with both inequalities reversed. Since $k(n)$ becomes infinite with n, it follows from these inequalities and (8) that with probability 1,

$$\lim_{n\to\infty} \frac{w_1 + \cdots + w_n}{n} = \alpha, \tag{15}$$

which completes the proof of the theorem.

If in Theorem 1 we let each x_n assume the values 0 and 1 with probabilities 1/2 and 1/2, then to each sequence $\{x_n\}$ corresponds the real number $0 \leqq x \leqq 1$ with dyadic expansion

$$x = .x_1x_2 \cdots x_n \cdots, \tag{16}$$

and probability in the space of sequences $\{x_n\}$ is identical with Lebesgue measure in the unit interval. Moreover, "product sequence" becomes "subsequence," and hence Theorem 1 specializes to the result referred to at the beginning of this note.

As another special case of Theorem 1 we shall derive a theorem on repetition sequences.

THEOREM 2. *Let* $\{a_n\}$ *be a sequence of real numbers summable* $(C, 1)$ *to the value* α *and such that* $\sum a_n^2/n^2 < \infty$. *To each* $0 \leqq t \leqq 1$ *with dyadic expansion*

$$t = .t_1 t_2 \cdots t_n \cdots \qquad (t_n = 0 \text{ or } 1) \tag{17}$$

let correspond the sequence $\{v_n\}$, *where* $v_1 = a_1$ *and*

$$v_{n+1} = a_{(1+t_1+\cdots+t_n)} \qquad (n = 1, 2, \cdots). \tag{18}$$

Then for almost every t *the sequence* $\{v_n\}$ *is summable* $(C, 1)$ *to the value* α.

PROOF. In Theorem 1 let each x_n take on all positive integral values, with $Pr(x=k) = 2^{-k}$ $(k = 1, 2, \cdots)$. Then

$$E(x_n) = \bar{x} = \sum_1^\infty k2^{-k} = 2; \quad E(x_n - \bar{x})^2 = \sum_1^\infty (k-2)^2 2^{-k} = 2. \tag{19}$$

Thus the hypotheses in Theorem 1 on the sequence $\{x_n\}$ are satisfied, and hence, with probability 1, the product sequence $\{w_n\}$ of $\{a_n\}$ by $\{x_n\}$ is summable $(C, 1)$ to the value α. Now to each t defined by (17) let correspond the sequence $\{x_n\}$ such that

(20) $x_1 = 1$ plus the number of consecutive 0's immediately following the decimal point in (17),

and for $n = 1, 2, \cdots$

(21) $x_{n+1} = 1$ plus the number of consecutive 0's immediately following the nth 1 in (17).

It is easy to verify that this correspondence $t \leftrightarrow \{x_n\}$ is one-to-one between the interval $0 \leqq t \leqq 1$ and the space of sequences $\{x_n\}$, and that it carries Lebesgue measure in the former into probability in the latter. Moreover, if $t \leftrightarrow \{x_n\}$ then the sequence $\{v_n\}$ associated with t by (18) is identical with the product $\{w_n\}$ of $\{a_n\}$ by $\{x_n\}$. Since with probability 1 the sequence $\{w_n\}$ is summable $(C, 1)$ to α, it follows that for almost every t the sequence $\{v_n\}$ is summable $(C, 1)$ to α, which was to be proved.

We conclude with an application of Theorem 2 to random sequences of transformations (compare [3]). Let $\{U_n\}$ $(n = 0, 1, 2, \cdots;$ $U_0 = I = \text{identity})$ be a sequence of transformations of a space M

into itself, let A be a subset of M, and let p be a point of M. Denote the characteristic function of A by

$$\phi_A(q) = \begin{matrix} 1 & \text{if} & q \in A, \\ 0 & \text{if} & q \notin A. \end{matrix} \tag{22}$$

If the sequence $\{\phi_A(U_n(p))\}$ is summable $(C, 1)$ to the value α let us say that the triple $(\{U_n\}, A, p)$ has the ergodic limit α. Let T be a fixed transformation of M into itself such that $(\{T^n\}, A, p)$ has the ergodic limit α, and to each t in the interval $0 \leqq t \leqq 1$ with dyadic expansion (17) let correspond the sequence of transformations $\{U_n\}$ where $U_0 = I$ and

$$U_n = \begin{matrix} TU_{n-1} & & \text{if} & t_n = 1 \\ IU_{n-1} = U_{n-1} & & \text{if} & t_n = 0 \end{matrix} \qquad (n = 1, 2, \cdots). \tag{23}$$

If we set $a_n = \phi_A(T^{n-1}(p))$ $(n = 1, 2, \cdots)$ and $v_n = \phi_A(U_{n-1}(p))$, then $v_1 = a_1$ and (18) holds. It follows at once from Theorem 2 that for almost every t the triple $(\{U_n\}, A, p)$ has the ergodic limit α.

If a completely additive measure μ is defined on a σ-field of subsets of M with $\mu(M) = 1$, and if the triple $(\{U_n\}, A, p)$ has the ergodic limit α except for a set of p with μ-measure 0, let us say that the pair $(\{U_n\}, A)$ has ergodic limit α. Assume that this holds for $(\{T^n\}, A)$. Consider the Cartesian product of M with the unit interval $0 \leqq t \leqq 1$, and let H denote the set of all pairs (p, t) for which the sequence $\{\phi_A(U_n(p))\}$ is summable $(C, 1)$ to α, where $\{U_n\}$ is defined by (23). By the result of the preceding paragraph the intersection of H with any fixed p-line (except for a set of p of μ-measure 0) has Lebesgue measure 1. If H is measurable it follows by Fubini's theorem that the intersection of H with any fixed t-line (except for a set of t of Lebesgue measure 0) has μ-measure 1. Hence for almost all t the pair $(\{U_n\}, A)$ has the ergodic limit α.

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