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ON A CONSTRUCTION FOR DIVISION ALGEBRAS OF ORDER 16

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It is not known whether there exist division algebras of order 16 (or greater) over the real number field $\mathfrak{R}$. In discussing the implications of this question in algebra and topology, A. A. Albert told the author that the well known Cayley-Dickson process[1] does not yield a division algebra of order 16 over $\mathfrak{R}$ and suggested a modification of that process which might. It is the purpose of this note to show that, while Albert's construction can in no instance yield such an algebra over $\mathfrak{R}$, it does yield division algebras of order 16 over other fields, in particular the rational number field R.

Initially consider an arbitrary field F. Let C be a Cayley-Dickson division algebra of order 8 over F. Define[2] an algebra of order 16 over F with elements $c=a+vb$, $z=x+vy$ (a, b, x, y in C) and with multiplication given by

$$cz = (a + vb)(x + vy) = (ax + g\cdot ybS) + v(aS\cdot y + xb) \tag{1}$$

where S is the involution $x\rightleftarrows xS=t(x)-x$ of C and g is some fixed element of C. The Cayley-Dickson process is of course the instance $g=\gamma$ in F.

For A to be a division algebra over F the right multiplication[1] R_z must be nonsingular for all $z\neq 0$ in A. Now

$$R_z = \begin{pmatrix} R_x & SR_y \\ SL_yL_g & L_x \end{pmatrix}$$

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[1] See [1] and [2] for background and notations. Numbers in brackets refer to the references cited at the end of the paper.

[2] We should remark that this modification of the Cayley-Dickson process does yield non-alternative division algebras of orders 4 and 8 over $\mathfrak{R}$ when applied to the algebras of complex numbers and real quaternions instead of to C. See R. H. Bruck, *Some results in the theory of linear non-associative algebras*, Trans. Amer. Math. Soc. vol. 56 (1944) pp. 141–199, Theorem 16C, Corollary 1, for a generalization.

and, if $g\neq 0$, R_z is nonsingular in case either $x=0$, $y\neq 0$ or $x\neq 0$, $y=0$. Therefore let $x\neq 0$, $y\neq 0$. Then

$$|R_z| = \begin{vmatrix} R_x & SR_y \\ SL_yL_g & L_x \end{vmatrix} = \begin{vmatrix} R_x & 0 \\ SL_yL_g & L_x - SL_yL_gR_x^{-1}SR_y \end{vmatrix}$$

$$= |R_x| \cdot \left| L_x - \frac{1}{n(x)} R_ySR_gSL_xR_y \right|$$

$$= |R_x| \cdot \left| L_x - \frac{1}{n(x)} R_ySR_gSR_x^{-1}R_{xy}L_x \right|$$

by a lemma of Moufang.[3] Hence $|R_z| = |L_x| \cdot |n(x)R_yR_g|^{-1} \cdot |n(x)R_xR_gR_y - n(g)n(y)R_{xy}|$. That is, A is a division algebra over F if and only if the transformation

$$n(x)R_xR_gR_y - n(g)n(y)R_{xy} \tag{2}$$

is nonsingular for all x, $y\neq 0$ in C.

Now let $F=\mathfrak{R}$, the field of real numbers. The non-scalar[4] element g generates $B\subset Q=B+uB$, Q a real quaternion algebra, $u^2=-n(u)$, $gu=u\cdot gS$. Multiplication in the Cayley algebra $C=Q+wQ$ is defined by the right multiplication

$$R_{q+wr} = R_{(q,r)} = \begin{pmatrix} R_q & SR_r \\ -SL_r & L_q \end{pmatrix}$$

for q, r in Q and S the involution $q\rightleftarrows qS=t(q)-q$ of Q. Specialize two elements x, y of C in the following manner. Let $y=u\in Q$; then $y^2=-n(y)$, $gy=y\cdot gS$. Let $x\in wQ$ and $n(x)=\zeta n(y)$ where $\zeta>0$, $\zeta^2=n(g)$. Then $xy=(0,x)R_{(y,0)}=(0,yx)$ and

$$|n(x)R_xR_gR_y - n(g)n(y)R_{xy}|$$

$$= \left| n(x)\begin{pmatrix} 0 & SR_x \\ -SL_x & 0 \end{pmatrix}\begin{pmatrix} R_g & 0 \\ 0 & L_g \end{pmatrix}\begin{pmatrix} R_y & 0 \\ 0 & L_y \end{pmatrix} - n(g)n(y)\begin{pmatrix} 0 & SR_{yx} \\ -SL_{yx} & 0 \end{pmatrix} \right|$$

$$= \begin{vmatrix} 0 & n(x)SR_xL_{yg} - n(g)n(y)SR_{yx} \\ n(g)n(y)SL_{yx} - n(x)SL_xR_{gy} & 0 \end{vmatrix}$$

$$= |R_xL_x| \cdot |n(x)L_{yg} - n(g)n(y)R_y| \cdot |n(g)n(y)L_y - n(x)R_{gy}|$$

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[3] [2, Lemma 1].

[4] The Cayley-Dickson process (the case $g=\gamma$, a scalar) may be eliminated by this argument too. If $\gamma\geqq 0$, let $y=\beta$ in $\mathfrak{R}$, $n(x)=\gamma\beta^2$; if $\gamma<0$, let $y=i$, $f=j$, $n(x)=-\gamma$ in what follows.

since x, y, g are quaternions. That is, $|n(x)L_{yg}-n(g)n(y)R_y|=0$ would imply that transformation (2) is singular.

Choose $f=\zeta y+yg$. Then $f\{n(x)L_{yg}-n(g)n(y)R_y\}=n(x)\zeta ygy+n(x)ygyg-n(g)n(y)\zeta y^2-n(g)n(y)ygy=n(x)y^2gS\cdot g-n(g)n(x)y^2=0$. Hence (2) is singular and A is not a division algebra over $\mathfrak{R}$.

The easy generalization that there is no choice of g to make A a division algebra of order 16 over any field F should not be made. For the singularity of transformation (2) implies that there exists an element $h\neq 0$ in C such that $n(x)\{(hx)g\}y=n(g)n(y)h(xy)$. Since the norm of a product is the product of the norms in an alternative division algebra,[5]

$$\overline{n(x)}^3n(h)n(g)n(y) = \overline{n(g)}^2\overline{n(y)}^3n(h)n(x) \quad \text{or} \quad \overline{n(x)}^2 = n(g)\overline{n(y)}^2$$

in case $g\neq 0$. That is, the transformation (2) cannot be singular (and A is therefore a division algebra) for any choice of g in C such that $n(g)$ is not the square of an element in F.

For example, let F be in particular the field R of rational numbers, and $g=1+i$ so that $n(g)=2$. Then the algebra A with multiplication defined by (1) is a division algebra of order 16 over R.

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[5] [2, Lemma 2].