TWO PROPERTIES OF THE FUNCTION COS x

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The function $f(x)=A\cos n(x+B)$, where A, B are any real constants and n is an integer, has the properties:

(I) $f(x)$ is real valued for all real x, of period 2π, and continuous.

(II) $f(x)$ is differentiable, and there exist constants a, b such that, for all x,

$$f'(x) = af(x+b).$$

(III) Given any constants a, a', b, b', there exist constants c, d such that, for all x,

$$af(x+a') + bf(x+b') = cf(x+d).$$

The object of this note is to show that, conversely, any function $f(x)$ which has property (I) and either (II) or (III) is necessarily of the form $f(x)=A\cos n(x+B)$. The latter result is used to derive the parallelogram law of addition of forces from certain other basic assumptions.

THEOREM 1. *Let $f(x)$ have properties* (I) *and* (II). *Then there exist constants A, B and an integer n such that $f(x)=A\cos n(x+B)$.*

PROOF. It follows from (II) that $f(x)$ is of class C^∞ and hence, from (I), can be represented by a convergent Fourier series, which, moreover, may be differentiated termwise. Thus for some complex constants k_n,

$$\begin{aligned} f(x) &= \sum k_n e^{inx}, \qquad f'(x) = \sum ink_n e^{inx}, \\ f'(x) - af(x+b) &= \sum k_n(in - ae^{inb})e^{inx}. \end{aligned} \tag{1}$$

It follows from (II) that for every integer n,

$$k_n(in - ae^{inb}) = 0. \tag{2}$$

If $f(x)\equiv 0$ then the theorem is trivial. Otherwise, there will exist an n for which $k_n \neq 0$. It follows that

$$in = ae^{inb}. \tag{3}$$

Taking absolute values we have

$$n = \pm a. \tag{4}$$

Thus there can be at most two values of n for which $k_n \neq 0$, and these

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values are negatives of one another. Thus for some integer n,

$$f(x) = k_{-n}e^{-inx} + k_n e^{inx}. \tag{5}$$

Since, by (I), $f(x)$ is real valued, it follows that k_{-n} and k_n are complex conjugates, and the proof is complete.

THEOREM 2. *Let $f(x)$ have properties* (I) *and* (III). *Then there exist constants A, B and an integer n such that $f(x) = A \cos n(x+B)$.*

PROOF. Since, by (I), $f(x)$ is continuous and of period 2π, it possesses at least a formal Fourier series,[1]

$$f(x) \sim \sum k_n e^{inx}, \qquad k_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt. \tag{6}$$

By (III), there exist constants c, d such that the function

$$g(x) = f(x+1) + f(x) - cf(x+d), \tag{7}$$

continuous and of period 2π, is identically zero. (This is the only consequence of (III) that we shall use.) Hence

$$0 \sim \sum k_n(e^{in} + 1 - ce^{ind})e^{inx}. \tag{8}$$

It follows that for every n,

$$k_n(e^{in} + 1 - ce^{ind}) = 0. \tag{9}$$

If $f(x) \equiv 0$ then the theorem is trivial. Otherwise, there will exist an n for which $k_n \neq 0$. For any such n it follows from (9) that

$$e^{in} + 1 = ce^{ind}. \tag{10}$$

Taking absolute values and squaring, it follows that

$$\cos n = (c^2 - 2)/2. \tag{11}$$

Hence if m and n are any two integers for which $k_m \cdot k_n \neq 0$, it follows from (11) that $\cos m = \cos n$. Hence for some integer r,

$$m = \pm n + 2\pi r. \tag{12}$$

Since π is irrational, it follows that $r = 0$ and $m = \pm n$. Thus the formal Fourier series for $f(x)$ consists of only two terms,

$$f(x) \sim k_{-n}e^{-inx} + k_n e^{inx}. \tag{13}$$

But in this case, since the functions e^{inx} are complete with respect

[1] The proof given here follows a suggestion of Paul R. Halmos. The author's original proof required the unnecessary assumption that $f(x)$ be of class C^1.

to continuous functions, the relation $\sim$ can be replaced by an identity,

$$f(x) = k_{-n}e^{-inx} + k_n e^{inx}. \tag{14}$$

Since $f(x)$ is real valued, k_{-n} and k_n must be complex conjugates, and the theorem is proved.

We shall now apply Theorem 2 to derive the law of addition of forces.[2] For simplicity, let us consider only forces acting at a fixed point in a fixed plane in which the angular coordinate x is defined. With such a force we identify the real valued function $F(x)$ which specifies the scalar component of the force in the direction x; thus a force is represented by a real valued function of period 2π. By the sum of two forces $F_1(x)$ and $F_2(x)$ we mean the function $F_1(x)+F_2(x)$. Our assumptions are the following.

(i) *All forces are geometrically similar.* By this we mean that there exists a fixed function $f(x)$ of period 2π such that any force $F(x)$ can be written in the form

$$F(x) = A_F \cdot f(x + \alpha_F), \tag{15}$$

where A_F and α_F are constants determined by $F(x)$. We need not assume that all values of the constants A_F and α_F can occur in (15), but we shall assume that there exist at least the forces $F_1(x)=f(x)$ and $F_2(x)=f(x+1)$.

(ii) *The sum of two forces is a force.* Together with (i), this implies that the function $f(x)$ has the property that for certain constants c, d and for every x,

$$f(x + 1) + f(x) = cf(x + d). \tag{16}$$

(iii) *The function $f(x)$ is continuous, non-constant, and vanishes for at most two values in the interval $0 \leqq x < 2\pi$.*

The proof of Theorem 2 shows that the function $f(x)$, continuous, real valued, of period 2π, and satisfying (16), must be of the form

$$f(x) = A \cos n(x + B), \tag{17}$$

where n is an integer. The hypotheses of (iii) ensure that n can be chosen as 1. The parallelogram law of addition of forces is an immediate consequence.

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[2] See G. Darboux, Bull. Sci. Math. vol. 9 (1875) pp. 281–288; also G. D. Birkhoff, Rice Institute Pamphlet vol. 28, no. 1 (1941) pp. 46–50.