ON THE ASYMPTOTIC EXPANSIONS OF ENTIRE FUNCTIONS DEFINED BY MACLAURIN SERIES

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1. Introduction. Let $f(z)$ be an entire function defined by a Maclaurin series of the form

$$\sum_{n=0}^{\infty} g(n)z^n \tag{1}$$

with infinite radius of convergence. We wish, in this paper, to make some observations of a general nature relative to the asymptotic behavior of this function in the neighborhood of the point at infinity. Some restrictions will, of course, be placed on the coefficient $g(n)$ of z^n.

To begin with, we shall suppose $g(n)$ to be of such character that we can replace n by the complex variable $w = x + iy$. Then we shall assume that the resulting function $g(w)$ satisfies the following two conditions when it is considered throughout any arbitrary right half-plane $x > x_0$:

(a) it is single-valued and analytic;

(b) corresponding to any positive number ϵ, there exists a positive constant K, dependent only on ϵ and x_0, such that

$$| g(x + iy) | < Ke^{(\gamma\pi+\epsilon)|y|} \tag{2}$$

where γ is a non-negative constant.

Under these conditions, if k is a positive integer such that $k \geqq \gamma$, then it is known that we may express $f(z)$ in the form

$$\begin{aligned} f(z) = \int_{-l-1/2}^{\infty} & g(x)[(-1)^{k+1}z]^x \frac{\sin k\pi x}{\sin \pi x}\, dx \\ & - \sum_{n=1}^{l} g(-n)z^{-n} + \xi(z, l), \end{aligned} \tag{3}$$

where l is any positive integer, and where if $|\arg [(-1)^{k+1}z]| < \pi$, we have $\lim_{|z|\to\infty} z^l \xi(z, l) = 0$.

This theorem was first proved for the special case in which $k = 1$ by W. B. Ford [1, p. 30].[1] The general form (3) is due to C. V.Newsom [4]. Both Ford and Newsom have used their results in finding the asymptotic expansions of several special types of functions [1, chap-

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[1] Numbers in square brackets refer to references at the end of the paper.

ters 6, 7; 5; 6]. The theorem was also used in [2]. A study of the examples appearing in these investigations has suggested to the writer the generalities which are noted in the following sections.

2. **A fundamental lemma.** It will be observed that equation (3) does not furnish complete information regarding the asymptotic expansions of the function $f(z)$, due to the presence of the integral in the right-hand member. By finding the asymptotic expansions of this integral under suitable restrictions on $g(w)$, we shall arrive at a general theorem concerning such expansions of $f(z)$.

The proof of the theorem rests upon a lemma regarding the asymptotic character of a certain definite integral. This lemma is as follows:

LEMMA. *Consider the function*

$$F(z) = \int_{-\lambda}^{\infty} H(x) z^x dx, \tag{4}$$

where λ *is a sufficiently large positive number.*[2] *Suppose that the function* $H(w)$, *where* $w=x+iy$, *is single-valued and analytic in the finite w-plane and, when considered throughout the arbitrary right half-plane* $x>x_0$, *can be expressed in the form*

$$H(w) = \frac{c_0}{\Gamma(w+1)} + \frac{c_1}{\Gamma(w+2)} + \cdots + \frac{c_s + \delta(w, s)}{\Gamma(w+s+1)},$$

where the c's are constants, and $\lim_{|w|\to\infty} \delta(w, s) = 0$; $s = 0, 1, 2, \cdots$. *Then for values of z of large modulus such that* $|\arg z| < \pi$, *we have*

$$F(z) \sim \begin{cases} e^z \sum_{n=0}^{\infty} c_n z^{-n}; & |\arg z| \leqq \pi/2, \\ 0; & |\arg z| > \pi/2. \end{cases} \tag{5}$$

The proof of this lemma is omitted here since it requires only slight changes in one of the discussions appearing in the work of Ford [1, art. 20].

3. **A general theorem.** We proceed now to study the function $f(z)$ defined by the series (1). We shall establish the following general theorem regarding its asymptotic expansions.

THEOREM. *Given the entire function*

[2] The result also holds if λ is a complex number whose real part is positive and sufficiently large. The integration then takes place from $-\lambda$ to $+\infty$ along the line parallel to the real axis.

$$(6)\qquad f(z) = \sum_{n=0}^{\infty} g(n)z^n,$$

suppose that the function $g(w)$, *where* $w=x+iy$, *is single-valued and analytic in the finite w-plane and, when considered throughout the arbitrary right half-plane* $x>x_0$, *can be expressed in the form*

$$(7)\quad g(w) = \sigma^w \left\{ \frac{c_0}{\Gamma(\alpha w + t)} + \frac{c_1}{\Gamma(\alpha w + t + 1)} + \cdots + \frac{c_s + \delta(\alpha w, s)}{\Gamma(\alpha w + t + s)} \right\},$$

where σ *and* α *are positive, the c's and t are constants, and* $\lim_{|w|\to\infty} \delta(\alpha w, s) = 0$; $s=0, 1, 2, \cdots$. *Then, when considered for values of z of large modulus such that* $-\pi < \arg z \leqq \pi$, *the function* $f(z)$ *is developable asymptotically in the form*

$$(8)\qquad f(z) \sim \frac{1}{\alpha} \sum_{\mu} \left\{ Y_\mu^{(1-t)} e^{Y_\mu} \sum_{n=0}^{\infty} c_n Y_\mu^{-n} \right\} - \sum_{n=1}^{\infty} g(-n) z^{-n},$$

where Y_μ *denotes the expression* $\sigma^{1/\alpha} z^{1/\alpha} e^{2\pi i\mu/\alpha}$, *and the symbol* $\sum_\mu$ *denotes summation over those integral values of* μ *which satisfy* $|\arg z + 2\pi\mu| \leqq \pi\alpha/2$.

Proof. Under the conditions postulated, when $x>x_0$, we may write

$$| g(x + iy) | < K e^{(\pi\alpha/2+\epsilon)|y|}$$

where ϵ and K have the meanings described in §1.[3] Consequently, the theorem of Ford and Newsom is applicable to the function $f(z)$ if k is properly chosen. If a proper choice be made, then, for the corresponding range of arg z, we have

$$(9)\qquad f(z) = \phi(z) - \sum_{n=1}^{l} g(-n)z^{-n} + \xi(z, l),$$

where $\phi(z)$ denotes the integral

$$\int_{-l-1/2}^{\infty} g(x) [(-1)^{k+1} z]^x \frac{\sin k\pi x}{\sin \pi x}\, dx.$$

In particular, if k is selected as the smallest odd integer $2p+1$ such that $k \geqq \alpha/2$, then the factor $[(-1)^{k+1}z]^x$ in the integrand of $\phi(z)$ becomes z^x, and (9) is then valid when $|\arg z| < \pi$. On the other hand, if k is taken as $2p+2$, then this factor becomes $(-z)^x$, and (9) holds for $0 < \arg z = \pi + \arg(-z) < 2\pi$, and hence in particular when $\arg z = \pi$.

[3] For the special property of the gamma function from which this inequality results, see [1, p. 61].

Now we have

$$\frac{\sin k\pi x}{\sin \pi x} = \begin{cases} \sum_{\mu=-p}^{p} e^{2\pi i\mu x}; & k = 2p+1, \\ e^{\pi i x}\sum_{\mu=-p-1}^{p} e^{2\pi i\mu x}; & k = 2p+2. \end{cases}$$

And since $e^{\pi ix}\cdot(-z)^x = z^x$, it follows that for either of the two values of k mentioned, the integral $\phi(z)$ is equal to the sum of k integrals of the form

$$(10) \qquad h_\mu(z) = \int_{-l-1/2}^{\infty} g(x)(z\cdot e^{2\pi i\mu})^x dx,$$

where we now have $-\pi < \arg z \leqq \pi$, and the integer μ takes on the values indicated in the above identities.

We shall next obtain the asymptotic expansions of the integral in (10). First, we recall that $g(x)$ is expressible in form (7). Hence, if we change the variable of integration in (10) from x to x' through the transformation $\alpha x + t = x' + 1$, and then drop the primes, (10) assumes the form

$$(11) \qquad h_\mu(z) = \frac{1}{\alpha} Y_\mu^{(1-t)} \int_{-\lambda}^{\infty} H(x) Y_\mu^x dx,$$

where Y_μ is as already defined, $\lambda = \alpha(l+1/2)+1-t$, and $H(x)$ is of the form

$$\frac{c_0}{\Gamma(x+1)} + \frac{c_1}{\Gamma(x+2)} + \cdots + \frac{c_s + \delta(x, s)}{\Gamma(x+s+1)}.$$

We may therefore apply the fundamental lemma of §2 to the integral appearing in (11), assuming that l has been chosen sufficiently large. In (5), z is to be replaced by Y_μ, and $\arg z$ by $(\arg z + 2\pi\mu)/\alpha$. Thus, for values of z such that $-\pi < \arg z \leqq \pi$ and $|\arg z + 2\pi\mu| < \pi\alpha$, we have

$$(12) \qquad h_\mu(z) \sim \begin{cases} \frac{1}{\alpha} Y_\mu^{(1-t)} e^{Y_\mu} \sum_{n=0}^{\infty} c_n (Y_\mu)^{-n}; \\ 0, \end{cases}$$

the first result holding when $|\arg z + 2\pi\mu| \leqq \pi\alpha/2$, and the second otherwise.

Having established (12), we have only to sum the expansions of

$h_\mu(z)$ over the proper values of μ in order to arrive at the expansion of the integral $\phi(z)$. Evidently this will yield the expansion

$$\phi(z) \sim \frac{1}{\alpha} \sum_\mu \left(Y_\mu^{(1-t)} e^{Y_\mu} \sum_{n=0}^{\infty} c_n Y_\mu^{-n} \right), \tag{13}$$

where $\sum_\mu$ has the meaning already stated.

Finally, if we replace $\phi(z)$ in (9) by its asymptotic expansion given in (13), and write the result in asymptotic notation, we arrive at (8). This conclude the proof of the theorem.

4. **Special cases. Dominant terms.** We shall now note the simplified forms assumed by (8) for certain values of α and of arg z. At the same time, we shall pick out the dominant terms in each form. First, if α is such that $0<\alpha<2$, then the condition $|\arg z+2\pi\mu| \leqq \pi\alpha/2$ is satisfied only by $\mu=0$ when $|\arg z| \leqq \pi\alpha/2$. Hence for such α and arg z, we have

$$f(z) \sim \frac{1}{\alpha} (\sigma z)^{(1-t)/\alpha} \exp (\sigma z)^{1/\alpha} \sum_{n=0}^{\infty} c_n(\sigma z)^{-n/\alpha} - \sum_{n=1}^{\infty} g(-n)z^{-n}. \tag{14}$$

Here we see that the first series is dominant when arg z is confined to the narrower range $|\arg z| <\pi\alpha/2$, but not when $|\arg z| =\pi\alpha/2$. Furthermore, if $\alpha=2$, and $|\arg z| <\pi$, expansion (14) still holds, and the first series is dominant.

On the other hand, if we have $0<\alpha<2$ and $|\arg z| >\pi\alpha/2$, but still such that $-\pi<\arg z\leqq\pi$, then no integral value of μ satisfies $|\arg z+2\pi\mu| \leqq\pi\alpha/2$. Hence, for such values of z, (8) reduces to the form

$$f(z) \sim - \sum_{n=1}^{\infty} g(-n)z^{-n}. \tag{15}$$

It is to be noted that this result is consistent with another theorem due to Ford [1, p. 4] according to which (15) holds throughout the open sector $\pi\alpha/2 <\arg z<(2-\alpha/2)\pi$.

Finally, we note that when $\alpha\geqq 2$, the terms affected by $\sum_\mu$ are dominant over the last series in (8), this being true for all values of arg z under consideration.

5. **Supplementary remarks.** Several supplementary observations relative to the theorem established in §3 are now to be noted. In the first place, we have required that the function $g(w)$ have no finite singularities. Suppose, however, that it does have a singularity at $w=w_0$, while (7) continues to hold for all values of $|w|$ sufficiently

large. Then if w_0 is not a negative integer, the theorem still holds provided that we subtract from the right member of (8) the asymptotic expansion of the loop integral

$$\frac{1}{2i}\int_C \frac{g(w)[(-1)^{k+1}z]^w}{e^{ki\pi w}\sin \pi w}\,dw. \tag{16}$$

The loop C surrounds the point w_0 and extends to infinity in any convenient direction lying in either the third or the fourth quadrant. If w_0 is a negative integer, say $-m$, then the term $g(-m)z^{-m}$ in (8) is suppressed, and the expansion of (16) is put in its place. For justification of these statements, the reader is referred to [3], where a method for finding the expansion of (16) is also discussed.

Secondly, the theorem which we have here established narrows down the difficulties involved in determining the asymptotic expansion of a given entire function of form (1). For the problem of finding the expansion of the integral appearing in (3) is solved if it can be shown that the function $g(w)$ can be written in the form (7).

Thirdly, the theorem can evidently be applied to a large variety of entire functions. In many instances in which the coefficient $g(n)$ involves the gamma function, relation (7) is satisfied. In this connection, we note that if the value of σ in (7) is unity, then $f(z)$ is necessarily the function

$$F_\alpha(z, t) = \sum_{n=0}^{\infty} \frac{h(n)}{\Gamma(\alpha n + t)} z^n$$

which was discussed in [2]. The writer plans to apply the theorem to certain further important types of functions, and to report the results in later papers.

References

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