57. P. M. Whitman: *Lattices and equivalence relations.* Preliminary report.

It is shown that any lattice is isomorphic to a sublattice of the lattice of all equivalence relations on some set. (Received January 28, 1944.)

ANALYSIS

58. Jesse Douglas: *Separable transformations with separable inverse.*

All transformations $X=f(x)+h(y)$, $Y=g(x)+k(y)$ are found whose inverses are of the same form. Six essentially different types are obtained. If x, y are interpreted as minimal coordinates $u+iv$, $u-iv$ (and X, Y similarly), we have all harmonic transformations whose inverses are harmonic. The paper will be published in full. (Received January 15, 1944.)

59. K. O. Friedrichs: *The identity of weak and strong extensions of differential operators.*

In applying the theory of linear operators in Hilbert spaces or spaces L_p to the solution of differential equation problems, it is impossible to retain the meaning of differentiation in the ordinary sense; the concept of differential operator must be extended. Two such extensions offer themselves, a "weak" and a "strong" one, that is, the adjoint of the "formal-adjoint" and essentially the closure. The purpose of the paper is to prove the identity of these two extensions for general linear differential operators. The main tool for the proof is a certain class of smoothening operators approximating unity. They yield the identity of both extensions immediately for differential operators with constant coefficients; they are a strong enough tool to yield this identity likewise for operators with non-constant coefficients. (Received December 3, 1943.)

60. B. M. Ingersoll: *On singularities of solutions of linear partial differential equations.*

Let $U(z, \bar{z})$, $z=x+iy$, $\bar{z}=x-iy$, x, y real, be a real solution of $L(U)\equiv\Delta U+AU_x+BU_y+CU=0$, where A, B, and C are entire functions when x and y are extended to complex values. To every such solution corresponds uniquely a complex solution $u(z, \bar{z})=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}z^m\bar{z}^n$ of $L(U)=0$, with the property that $\sum_{n=0}^{\infty}A_{0n}\bar{z}^n=\pi U(0, 0)$ $\exp\left(-\int_0^{\bar{z}} a(0, \bar{z})d\bar{z}\right)$, where $a(z, \bar{z})\equiv(1/4)\{A[((z+\bar{z})/2, (z-\bar{z})/2i)]+iB[((z+\bar{z})/2, (z-\bar{z})/2i)]\}$. These solutions were introduced by Bergman (Rec. Math. (Mat. Sbornik) N. S. vol. 2 pp. 1169–1198 and Trans. Amer. Math. Soc. vol. 53 pp. 130–155) who showed that the location of the singularities of $u(z, \bar{z})$ is determined by the sequence $\{A_{m0}\}$. Employing this result the author investigates the relations between sequences $\{A_{mk}\}$, k fixed, $m=0, 1, 2, \cdots$, and the positions of singularities of $u(z, \bar{z})$. For example, using a result of Mandelbrojt (C. R. Acad. Sci. Paris, 1937, pp. 1456–1458) he determines the arguments of the singularities on the circle of convergence of $u(z, \bar{z})$ in terms of the sequence $\{A_{mk}\}$, k fixed. In the last section of the paper, using explicitly an integral representation of the complex solutions $u(z, \bar{z})$, the author investigates the real solutions $U(z, \bar{z})=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}D_{mn}z^m\bar{z}^n$ of $L(U)=0$. He constructs, in terms of $\{D_{mk}\}$, k fixed, $m=0, 1, 2, \cdots$, and some of the derivatives of the coefficients of L, an analytic function $f(z/2)$ whose singularities coincide with those of $U(z, \bar{z})$. (Received January 28, 1944.)

61. Mark Kac: *On real zeros of some functions.*

The purpose of this note is to call attention to some applications of the formula for the number of real roots given recently by the author (*On the distribution of values of trigonometric sums with linearly independent frequencies*, Amer. J. Math. vol. 65 (1943) pp. 609–615, in particular §3). If $f(x)$ has a continuous first derivative and a finite number of turning points in (a, b) then the number of zeros of $f(x)$ in (a, b) is given by the formula $(\pi/2)\int_{-\infty}^{+\infty}\{\int_a^b \cos(\xi f(x))|f'(x)|dx\}d\xi$, with the understanding that multiple roots are counted only once and that end points (a and b) if they happen to be roots are counted as 1/2 each. This formula is a source of amusing identities. For instance if $f(x)=\cos x$, $a=0$, $b=2\pi$ one obtains easily that $4\pi^2=\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty}((J_0(\xi)-J_0(\xi^2+\eta^2)^{1/2})/\eta^2)d\xi d\eta$. Also this formula is very convenient in computing average numbers of real roots of certain random functions. (Received January 31, 1944.)

62. A. T. Lonseth: *Dirichlet's principle for $\Delta u=F_u(u; x, y)$.*

This nonlinear elliptic partial differential equation (Δ = Laplacean operator) is the Euler-Lagrange equation for the minimization of $\int\int\{u_x^2+u_y^2+2F(u; x, y)\}dx\,dy$. Here the simultaneous boundary-value and minimum problem solution is obtained by modifying Courant's method for self-adjoint linear elliptic equations (R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. II, Berlin, 1937, chap. 7). It is assumed that $F(u; x, y)$ is of class C'' and convex in u, and of class C' in x and y. Assumptions as to region and boundary values are as in Courant-Hilbert. (Received January 17, 1944.)

63. Szolem Mandelbrojt: *Some theorems connected with the theory of infinitely differentiable functions.*

The author gives a simple and elementary proof of the necessity of a well known necessary and sufficient condition for quasi analyticity, and an immediate proof of a theorem concerning Watson's problem. The author proves also an inverse to a theorem of S. Bernstein on the best approximation on the whole real axis. (Received January 27, 1944.)

64. H. E. Newell: *The solutions of a certain linear matric differential equation.*

The linear matric differential equation $dY(x, \lambda)/dx=\{\lambda(\delta_{ij}r_j(x))+(q_{ij}(x, \lambda))\}\cdot Y(x, \lambda)$, x and λ being complex, has been shown to possess under certain conditions solutions of the form $P(x, \lambda)E(x, \lambda)$, where $E(x, \lambda)=(\delta_{ij}\exp\{\lambda\int^x r_j(x)dx\})$ and $P(x, \lambda)$, analytic in x, reduces uniformly in x to the identity matrix when λ becomes infinite. The theory, originally developed by R. E. Langer (R. E. Langer, *The boundary problem of an ordinary linear differential system in the complex domain*, Trans. Amer. Math. Soc. vol. 46 (1939) pp. 151–162; correction, p. 467) and later extended by the author (H. E. Newell, Jr., *The asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain*, Duke Math. J. vol. 9 (1942) pp. 245–258, and vol. 10 (1943) pp. 705–709), was applied for the most part to cases in which the elements $q_{ij}(x, \lambda)$ were analytic and bounded. The present paper treats of certain special cases in which the functions $q_{ij}(x, \lambda)$ may have poles on the boundary of the x region in question. (Received January 21, 1944.)

65. A. M. Peiser: *Some applications of Fourier analysis to the study of real roots of algebraic equations.*

Let $N_n(x) = N_n(x_0, \cdots, x_n)$ denote the number of real roots of $F \equiv \sum_{j=0}^{n} x_j t^j = 0$. It is shown that at each continuity point of N_n, $N_n(x) = \lim_{h \to \infty} (h/(2\pi)^{1/2}) \int_{-\infty}^{\infty} A(t) \cdot | Y_n(x, t) | \exp [-h^2 X_n^2(x, t)/2] dt$ where $A(t)$ is a known positive function of t alone, and X_n and Y_n are orthogonal, normal linear forms in $x_0, \cdots, x_n$. Thus, if the x's are independent, normally distributed random variables, then X_n and Y_n are normally distributed and, because of the orthogonality, independent. For the average number (mathematical expectation = m.e.) of real roots of $F = 0$ in this case we find m.e. $\{N_n(x)\} = \lim_{h \to \infty} (h/(2\pi)^{1/2}) \int_{-\infty}^{\infty} A(t)$ m.e. $\{ | Y_n(x, t) | \}$ m.e. $\{ \exp [-h^2 X_n^2(x, t)/2] \} dt$. Suppose now that m.e. $\{x_j\} = c_j$ and that for each j, m.e. $\{x_j^2\} = s$. The author compares the number of real roots $N_n(c)$ of $F_1 \equiv \sum_{j=0}^{n} c_j t^j = 0$ with the number of real roots $N_n(c+sx)$ of $F_2 \equiv \sum_{j=0}^{n} (c_j + s x_j) t^j = 0$, and shows how to choose s so that m.e. $\{N_n(c+sx)\}$ is near $N_n(c)$. This may have an application to empirical equations, for we can consider $F_1 = 0$ as the theoretical, or "true" equation, and $F_2 = 0$ as an "observed" equation in which each coefficient is measured with precision s. (Received January 28, 1944.)

66. Harry Pollard: *Fourier series with coefficients in a Banach space.*

Let $f(t)$ be a function on $(0, 1)$ to the complex Banach space B. Bochner has shown that a large part of the older theory of Fourier series carries over to functions of this character, but breaks down in the fundamental L^2 theory. In particular Bessel's inequality is no longer valid. It is proved in this paper that the inequality carries over if and only if B admits a scalar product. (Received January 24, 1944.)

67. C. E. Rickart: *Representation of linear transformations on summable functions.*

This paper contains an integral representation theorem for the general bounded linear transformation on summable functions to an arbitrary Banach space. The representation is obtained by means of an abstract Radon-Nikodym theorem proved earlier by the author (see abstract 49-11-270). (Received January 26, 1944.)

68. H. E. Robbins: *A note on the Riemann integral.*

Let $f(x)$ be a continuous function. If, in the usual definition of the Riemann integral of $f(x)$ from a to b as the limit of S_n equals the sum of terms of the form $f(\bar{x}_i)(x_i - x_{i-1})$, the points of subdivision, $x_0 = a, x_1, \cdots, x_n = b$, are *not* assumed to have the property $x_i \geqq x_{i-1}$, then S_n need not tend to a limit as $d_n = \max | x_i - x_{i-1} |$ tends to 0. But if any constant $C \geqq (b-a)$ is given in advance and if the points of every subdivision satisfy the inequality, sum of $| x_i - x_{i-1} | \leqq C$, then S_n will tend to the Riemann integral of $f(x)$ as d_n tends to 0. (Received January 28, 1944.)

69. F. H. Safford: *Analysis of a non-harmonic wave.*

The present paper considers the problem of expressing a periodic function by means of sines and cosines up to p harmonics, when the values of the function are given as ordinates for equidistant abscissas, with no further information. This is a variation of the method of selected ordinates used by J. Fischer-Hinnen in 1901, but with p coefficients for the sine terms and $p+1$ coefficients for the cosine terms, corresponding terms being combined into the form $A \sin n(t+\theta)$. The period may be taken as 2π, and the values of n are from 0 to p. It is necessary to use a second set of ordinates at the distance $(\pi/2p)$ from the first set thus providing $2p+1$ disposable constants corresponding to the same number of points from the observed function. The presence of higher harmonics with composite indices is duly provided for. The theorem concerning the sums of equidistant ordinates of sine curves provides the components of the observed ordinates and the latter are linearly related to the disposable constants. Thus for each harmonic it is possible to obtain its amplitude and phase angle without trigonometric ambiguity, in terms of the observed values of the given ordinates. (Received January 24, 1944.)

70. Raphael Salem: *On a theorem of Bohr and Pál.*

The paper gives a short and simple proof of the following extension of a theorem of Pál, due to H. Bohr: given a function $\phi(t)$ continuous and of period 2π, there exists a function $t(\theta)$ $(t(0)=0,\ t(2\pi)=2\pi)$, continuous, strictly increasing, such that the Fourier series of $\phi(t(\theta))$ converges uniformly for $0\leqq\theta\leqq 2\pi$. (Received December 29, 1943.)

71. Raphael Salem: *Sets of uniqueness and sets of multiplicity.* II.

This paper is a continuation of a paper which appeared under the same title in Trans. Amer. Math. Soc. vol. 54 (1943) pp. 218–228. It gives the necessary and sufficient condition for an unsymmetric perfect set of constant ratio to be a set of uniqueness. It deals also with some aspects of the classification in sets of uniqueness and sets of multiplicity of symmetrical perfect sets of the Cantor type and of variable ratio of dissection. (Received December 29, 1943.)

72. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* IV.

Whenever an iterative compositional functional equation has a solution containing an arbitrary function in two or more variables, the possibility of an associated representation of substitution groups by functional equations may be entertained. A simple example is $f(u, x_2, \cdots, x_n, v)=f(x_1, x_2, \cdots, x_{n+1})$ where $u=f(x_1, t_1, t_2, \cdots, t_n)$ and $v=f(x_{n+1}, t_1, t_2, \cdots, t_n)(n=1, 2, \cdots)$ with solution: $f(x_1, x_2, \cdots, x_{n+1})=x_1-x_{n+1}+\alpha(x_2, \cdots, x_n)$. A group of equations follows if $f(x_1, x_2, \cdots, x_{n+1})=f(x_1, x_{i_2}, \cdots, x_{i_n}, x_{n+1})$. In this and certain analogous representations of a substitution group G, the equation corresponding to the identical substitution has a solution containing an arbitrary function and the corresponding group of equations has a solution based on the invariance of the latter function under G. A generalized associative equation as inverse of the above equation corresponds to the identical element in the group of equations: $\phi\{\phi(x_1, x_2, \cdots, x_n, y_1), y_2, \cdots, y_n, y_{n+1}\}=\phi\{x_1, x_2, \cdots, x_n, \phi(y_1, y_{i_2}, \cdots, y_{i_n}, y_{n+1})\}$. Another example is based on $f(u, u, \cdots, u)=u$, where $u=f(x_1, x_2, \cdots, x_{n+1})$ with solution: $f(x_1, x_2, \cdots, x_{n+1})=\alpha(w_1, w_2, \cdots, w_n)+x_{n+1}$ where $w_i=x_i-x_{n+1}$ and α is arbitrary except for $\alpha(0, 0, \cdots, 0)=0$. (Received January 24, 1944.)

73. A. R. Schweitzer: *On functional equations with solutions containing arbitrary functions.* V.

The quasi-transitive equation, $f(u_1, u_2, \cdots, u_{n+1})=f(x_1, x_2, \cdots, x_{n+1})$,

$u_i = f(x_i, t_1, t_2, \cdots, t_n)$, is made the basis of an $(n+1)$-adic genesis of the group concept by postulating further: (1) Given $f(x_1, x_2, \cdots, x_{n+1})$, there exists $\phi(x_1, x_2, \cdots, x_{n+1})$ such that $f\{\phi(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\} = x$ and $\phi\{f(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\} = x$. (2) If $x \neq y$, then $f(x, x, \cdots, x) = f(y, y, \cdots, y)$. (3) The set S of elements x_i is closed under the compositions f and ϕ. The preceding postulates define a group under binary composition if in the functions $f(x_1, x_2, \cdots, x_{n+1})$ and $\phi(x_1, x_2, \cdots, x_{n+1})$, $x_2 = x_3 = \cdots = x_{n+1}$. The postulates are also satisfied in the domain of abstract groups under $(n+1)$-ary composition if $f(x_1, x_2, \cdots, x_{n+1}) = x_1 \cdot x_{n+1}^{-1} \cdot \alpha(w_2, w_3, \cdots, w_n)$ where $w_i = x_i \cdot x_{n+1}^{-1}$ and α is an arbitrary "function" of group elements. Finally, the postulates are satisfied by a set of elements closed under the binary compositions of a commutative group and any one of a class of specialized operations represented by the function α, including multiplication of arguments. An analogous genesis based on a generalized associative equation is discussed. (Received January 24, 1944.)

74. M. F. Smiley: *An extension of metric distributive lattices with an application in general analysis.*

It is shown that a metric distributive lattice (Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, 1940, pp. 41 and 74) may be embedded in a field of sets with a finitely additive measure provided that the modular functional of the basic lattice is bounded from below. Under the additional hypothesis that this functional is bounded, it is proved that the corresponding measure is *completely additive*. This result is applied to E. H. Moore's last instance of his second general analysis theory (*General analysis*, Part I, Memoirs of the American Philosophical Society, vol. 1, 1935, pp. 14–15). In this example the positive character of Moore's basic matrix is found to depend essentially on the distributive law. The proof of the major result employs theorems of H. Wallman (*Lattices and topological spaces*, Ann. of Math. (2) vol. 39 (1938) pp. 112–126) and of M. H. Stone (*Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 376–481) on the representation of Boolean rings, as well as methods of H. M. MacNeille (*Extension of a distributive lattice to a Boolean ring*, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 452–455) and of S. Kakutani (*Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. (2) vol. 42 (1941) p. 533). (Received January 10, 1944.)

75. F. T. Wang: *On Riesz summability of Fourier series.* IV.

Let $f(t)$ be an integrable periodic function with period 2π and $\phi_\alpha(t)$ be the fractional integral of $\phi(t) = 2^{-1}\{f(x+t)+f(x-t)-2s\}$ of order α. Then we have the following results: (1) If $\phi_\alpha(t) = o(t^\alpha/\log t^{-1})$, $\alpha > 0$, as $t \to 0$, then the Fourier series of $f(t)$ is summable by Riesz typical mean of type $\lambda_n = \exp(\log n)^{1+\alpha^{-1}}$ and of order $\alpha+1$, or simply, summable $\exp(\log n)^{1+\alpha^{-1}}$, $\alpha+1$, to sum s at $t = x$. (2) If $\sum_{n=2}^{\infty}(a_n^2+b_n^2)(\log n)^\alpha$ $(0<\alpha<1)$ converges then the series $\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is summable $\exp(\log n)^{\alpha+1}$, δ, $\delta > 0$, almost everywhere. (Received January 5, 1944.)

76. F. T. Wang: *Some remarks on oscillating series.*

Let $\sigma_n^{(r)}$ be the rth Cesàro mean of the series $\sum_{n=0}^{\infty} a_n$ where r is a positive integer. Then we have the following result: If $\sigma_n^{(r)} - s = o(n^{-r\alpha})$, $0<\alpha<1$, as $n \to \infty$ and $a_n > -Kn^{\alpha-1}$, then the series $\sum_{n=0}^{\infty} a_n$ converges to sum s. For the case $r=1$ this has been proved by Boas. The author proves in this note the result for $r=2$, and gives an example from the theory of Fourier series that there exists a divergent series $\sum_{n=0} a_n$ such that $\sigma_n^{(2)} - s = 0(n^{-2\alpha'})$, $0 < \alpha' < \alpha < 1$ and $a_n = O(n^{\alpha-1})$. (Received January 5, 1944.)

77. D. V. Widder: *The iterates of the Laplace kernel.*

The iterates of the Laplace kernel $G_0(x, y) = e^{-xy}$ are defined by the recursion relation $G_n(x, y) = \int_0^\infty G_0(x, t)G_{n-1}(t, y)dt$, $n = 1, 2, \cdots$. In an earlier paper (Bull. Amer. Soc. vol. 43 p. 813) the author determined explicitly all these functions which have odd subscripts. They are rational functions of x, y and $\log (x/y)$. In the present paper the iterates with even subscripts are studied. They cannot be expressed in terms of the elementary functions. Complete asymptotic series for their behavior near $x = +\infty$ are obtained. For their behavior near the origin they are developed in power series which converge for all positive values of x. The method consists in expressing the nth iterate in terms of a function $h_n(x)$ whose bilateral Laplace transform turns out to be $\Gamma(-s)^{n+1}\,\Gamma(s+1)^n$. The function $h_n(x)$ is in turn expressible in terms of a new set of transcendental functions related to the familiar exponential integral $EI(x)$. (Received January 14, 1944.)

Applied Mathematics

78. Stefan Bergman: *On solutions of certain partial differential equations in three variables.*

(I) Let $E(x, y, z, t, \tau)$ be a solution of the equation $G(E) \equiv L(E) + [\tau^{-1}U^{-1}(1-\tau^2)(E_x + iE_y \cos t + iE_z \sin t)_\tau] + A[(1/2)\tau^{-1}(1-\tau^2)E]_\tau$ which satisfies certain boundary conditions. Here $L(E) \equiv \Delta E + A(xE_x + yE_y + zE_z) + CE$, $U \equiv x + iy \cos t + iz \sin t$. Then $\psi(x, y, z) = \int_0^{2\pi}\int_{-1}^{1} Ef[(1/2)u(1-\tau^2), t]/dtd\tau$, where $f(\zeta, t)$ is an arbitrary analytic function of ζ and t, will be a solution of $L(\psi) = 0$. (II) If A and C are entire functions of $r^2 = x^2 + y^2 + z^2$ alone, then $G(E)$ becomes $G_0(E) \equiv (1-\tau^2)E_{\tau\tau} - \tau^{-1}(1+\tau^2)E_\tau + r\tau[E_{rr} + 2r^{-1}E_r + BE] = 0$ where $B = [-(3/2)A - (1/2)rA_r + C - (1/4)r^2A^2]$. The author shows that in the case II, there always exists a solution $E = H(r, \tau)$ of $G_0(E) = 0$, which is an entire function of r. The author considers vectors $S = (\psi_1, \psi_2, \psi_3)$ where $\psi_1 = \int_0^{2\pi}\int_{-1}^{1} Hfdtd\tau$, $\psi_2 = i\int_0^{2\pi}\int_{-1}^{1} Hf \cos tdtd\tau$, $\psi_3 = i\int_0^{2\pi}\int_{-1}^{1} Hf \sin tdtd\tau$. Clearly $L(\psi_K) = 0$, $K = 1, 2, 3$. Let c^1 be a simple closed curve which lies on a sphere $x^2 + y^2 + z^2 = \text{const}$. If the ψ_K are regular in this sphere then $\int_{c^1} S \cdot dX = 0$. Here $X = (x, y, z)$, and $\cdot$ means the interior product. "Residue" theorems are derived if the ψ_K have singularities in the above sphere. Applications in the theory of waves propagation are indicated. (Received January 28, 1944.)

79. Nathaniel Coburn: *A boundary value problem in plane plasticity for the Coulomb yield condition.*

The following problem is studied: given a semi-plane $x > 0$, composed of plastic material which follows the Coulomb yield condition; the stresses σ_x, σ_y, σ_{xy} acting on the boundary $x = 0$ are considered as known; to find the stresses at any point in the interior of the semi-plane. The method of attack is a modification of that used in studying a similar problem for a perfectly plastic material. The stresses and the functions $\sin 2\gamma$, $\cos 2\gamma$ (where γ is the angle between the x-axis and a tangent to a line of principal shearing stress) are expanded in power series of the friction coefficient. Substituting these power series into the Levy equations, there results an infinite set of Levy equations for the various approximations to the stresses. By requiring that