integrodifferential and difference equations it solves. It has a wealth of illustrative examples, done in the text, and many problems for the student at the ends of the chapters (no answers). There is a very extensive table of Laplace transforms, as useful as a table of integrals in a calculus course. In chapter 1 and in appendix B a valuable comparison of the Laplace method with other possible techniques is given. Historical notes on the mathematical theory appear in appendix C. Finally there is one of the most extensive bibliographies on the subject yet to appear.

D. V. WIDDER

A treatise on projective differential geometry. By Ernest Preston Lane. Chicago, University of Chicago Press, 1942. 9+466 pp. $6.00.

Since the appearance of the author's earlier volume, *Projective differential geometry of curves and surfaces* (University of Chicago Press, Chicago, 1932), significant contributions to the field of projective differential geometry have been made by geometers in various parts of the world. In the preface the author does not claim that the present treatise is exhaustive, but states that it represents the fruit of ten years of study and investigation and gives an account of the author's experience with those portions of the subject which interested him most. It is with respect to those portions of the subject, then, that the reviewer interprets the author's statement, appearing earlier in the paragraph, that "the present volume integrates the new material with the old and gives a connected exposition of the theory to date." The treatise reports results of studies made by many workers in the field, and expounds a wide range of topics. It is also notable, however, that some of the newer topics that have attracted rather general interest have not been mentioned. Parts of the proofs, involving calculations that are difficult for the uninitiated, are so frequently left to the care of the reader that only the more advanced students of the subject can use the volume successfully as a textbook. The treatise is properly designed to serve as a reference book for the research worker in the field. Geometric concepts and results are consistently described in a lucid graphic manner.

It is satisfying to observe that the present volume devotes considerably more attention to the methods of Wilczynski on the study of curves by means of linear differential equations than does the earlier volume. It is disappointing, however, to find no mention made of Stouffer's simplifications of Wilczynski's methods of determining (i) canonical power series expansions for the local equations of plane and space curves, and (ii) the geometric characterizations of the vertices of the associated reference triangles and tetrahedra. About three times as much space is afforded the theories of curves as was allotted to this subject in the earlier volume. A similarly expanded amount of space is afforded the subject of surfaces in the present volume. The metric and affine applications are omitted. The Fubini method of the use of differential forms is omitted. The theorems stated in the earlier volume as exercises appear without proofs at the ends of the sections as supplementary theorems. Topics in the study of conjugate nets are treated much more fully than in the former volume. In particular, three types of sequences of Laplace are treated extensively in the present volume, whereas the subject of sequences of Laplace received scant attention in the earlier volume. A twenty page chapter is devoted to the study of plane nets. Grove's studies of the neighborhoods of singular points of surfaces are not mentioned. Their omission represents a distinct loss. A brief treatment of Fubini's theory of W-congruences is given. This theory could have been brought more nearly up to date had the contributions of Jonas (1937) and Fubini (1940) been added.

The subjects of plane and space curves in hyper- and ordinary space are treated in the first three chapters. The next three chapters deal with surfaces in ordinary projective space. Two chapters are devoted to the study of conjugate nets and one is devoted to the theory of plane nets. Transformations of surfaces and surfaces and varieties in hyperspace are studied in the last two chapters. The exposition is uniformly excellent throughout the volume. No attempt will be made to review each chapter by topics, but to give an indication of the nature of the relationship of the material included in the treatise to other material in the literature a topical review is here given of the subject matter contained in the two chapters on surfaces in ordinary space.

Chapters V and VI contain expositions of fundamental topics in the theory of analytic surfaces in ordinary three-dimensional space. The parametric curves on each surface under consideration have been selected to be the asymptotic curves. The form of the fundamental system of defining differential equations is deduced from a consideration of the curvilinear differential equation of the asymptotic curves of a surface. Integrability conditions are discussed and a study is made of the effects on the differential equations of the most general transformations of variables which leave the surface undisturbed and preserve the parametric character of the asymptotic curves. Use is then made of special transformations of proportionality factor, from original homogeneous coordinates x to new coordinates $\bar{x}$, to obtain reductions of the system of differential equations to canonical forms. The first canonical form results from the choice of coordinates $\bar{x}_i$ defined by $\bar{x}_i = x_i/x_4$. The second and third canonical forms are Wilzynski's and Fubini's, respectively. Both are obtained by making a transformation of the form $x_i = \lambda\bar{x}_i$. For Wilczynski's canonical form $\lambda = e^{\theta/2}$, and for Fubini's canonical form $\lambda = (e^\theta/\beta\gamma)^{1/2}$, wherein θ, β, γ are coefficients of the original differential equations of the surface. The canonical forms of Wilczynski and Fubini have been used extensively by various geometers. In order to arrive at Fubini's canonical form it is necessary that the surface under consideration be not ruled. No mention is made in the present volume of Grove's important class of canonical forms. This class not only contains Fubini's canonical form but also contains other canonical forms of similar geometric interest which, in marked contrast with Fubini's canonical form, exist whether or not the surface is ruled. A brief but interesting account is given in the present treatise of a method of calculation of an expansion which represents one nonhomogeneous coordinate z as a power-series in the two nonhomogeneous coordinates x and y of a point X of a surface. A great expenditure of labor is involved in the calculation of the terms of the fifth and sixth degrees of this power-series. The theory and method of procedure are adequately explained; the tedious details of the calculations are appropriately omitted. The author has contributed much to the development of such power-series expansions. A canonical form for the differential equations of a ruled surface is obtained in a direct and simple manner. The relations between the coefficients of these equations and those on which Wilczynski based his theory of ruled surfaces are then established. At this point properties of the curves of sections by planes through an asymptotic tangent are deduced. The flecnode curves are briefly considered. The quadrics of Darboux are defined and their equations with reference to a local coordinate system are obtained. The quadric of Lie and the asymptotic osculating quadrics of Bompiani are also defined and their local equations are calculated. Bompiani's characterization of the quadric of Wilczynski as a special asymptotic osculating quadric is deduced. A somewhat similar characterization of the quadric of Fubini is reached. Reciprocal congruences are defined and studied, and their developables and focal surfaces are determined. Although G. M. Green pioneered in this subject of reciprocal congruences by his study of congruences in the relation R, no reference is made to Green's work. The author's presentation appears to be essentially an interpretation of Green's work in terms of a different system of differential equations than was employed by Green. The equations and expressions are simpler in the forms in which they were originally expressed by Green, in terms of the coefficients of Wilczynski's canonical form. Considerable attention is given to the geometric characterization of various canonical lines, but no reference is made to a known unified geometric characterization of the canonical pencil.

Chapter VI opens with the introduction of the point-to-plane correspondence in which each point of a surface corresponds to the tangent plane to the surface at the point. This correspondence has been thoroughly studied by the author and Fubini and Čech. The axis, ray, associate axis, and associate ray are defined at a point in connection with a conjugate net, and relations among these are established. The author's studies of a general pencil of conjugate nets and the associated ray-point cubic and ray-conic are presented. The cones which correspond to the ray-point cubic and the ray-conic by the point-to-plane correspondence mentioned above are determined. These are the axis plane cone and the axis quadric cone, respectively. Hypergeodesics are defined on a surface by means of a curvilinear differential equation of the second order. Several properties of hypergeodesics are enumerated. A cone of the third class is determined as the envelope at a point of the surface of all of the osculating planes of the hypergeodesics of a family passing through that point. The cone has three cusp planes which intersect in a line called the cusp-axis of the system of hypergeodesics at the point. It is shown that the curves of a pencil of conjugate nets on a surface form a family of hypergeodesics. Other families of hypergeodesics are discussed, among which are the projective geodesics and the union curves of a congruence Γ_1 on a surface. The projective geodesics are the extremal curves of the invariant integral

$$\int_{u_0}^{u} (\beta\gamma v')^{1/2} du$$

known as the projective arc length of the curve along which it is calculated.

The property of the projective normal which Fubini discovered is established, namely, at a point on a surface the cusp-axis of the projective geodesics is the projective normal. The quadric of Moutard for a direction at a point of a surface is defined and its local equation is calculated. The author describes some properties of the transformations of Čech and presents his own construction for the point P_k which corresponds to a given plane in the general transformation of

Čech at a point P_x of the surface S. The author fails to mention the generalization of this transformation by Fubini and Čech. A simple construction is known for the elements which correspond in this more general transformation. The pangeodesics are defined as the extremal curves of the integral invariant

$$\int_{u_0}^{u}\left(\frac{\beta+\gamma v'^3}{v'}\right)du.$$

The geometric characterizations of the pangeodesics obtained by Lane and Segre appear in the volume. Formulas for the differentiation of local point coordinates are deduced and it is proved that with the use of these formulas the ordinary theory of envelopes may be applied in finding the envelope of a family of surfaces S_1 defined by $f(x_1, x_2, x_3, x_4, u, v)=0$ in which $x_1, \cdots, x_4$ are the local coordinates of a variable point on the surface S_1 and u, v are the curvilinear coordinates of the point x on the surface S. An application of this theory of envelopes is given in the determination of the local coordinates of the vertices of the tetrahedron of Demoulin. Results of L. Green, B. Segre, Su, and Thomsen and Mayer, on the respective subjects, the envelope of the quadric of Moutard, projective curves, projectively minimal curves, and projectively minimal surfaces, are reported as supplementary theorems. Important concepts which might have been included are those of projective curvatures and torsions of a curve on a surface, with respect to differential forms of the surface. New characterizations of the curves of Darboux, the curves of Segre, and of the Darboux-Segre pencil of conjugate nets could have been given. Also simple geometric characterizations of the important integral invariants

$$\int_{u_0}^{u}(\beta\gamma v')^{1/2}du, \qquad \int_{u_0}^{u}\left(\frac{\beta+\gamma v'^3}{v'}\right)du,$$

are known, but none are given.

The author has on former occasions called the attention of differential geometers to the need of an intrinsic tensor calculus for the study of a variety V_k immersed in a projective space S_n, where $n>k>1$ and $k\neq n-1$. In the present treatise he has neglected to announce that Grove has recently developed such a tensor calculus. Grove has shown that certain tensors arising in the theory of the geometry of paths on a variety V_k can be expressed in terms of tensors arising in the study of the variety from the point of view of classical projective differential geometry. The works of E. Cartan on projective differential geometry have also been overlooked. Cartan has used his method of moving reference systems with outstanding effectiveness to study both classical projective differential geometry and the generalized projective differential geometry of spaces with a projective connection.

To form a fair estimate of the treatise, in view of numerous omissions of important topics, we must keep in mind that the aim of the author was not to write an exhaustive treatise on the subject but to present an account of his experience with those portions of the subject which interested him most.

The style of the book is attractive, the typography is clear and restful to the eye, and the proofreading has been carefully done. The research worker will find the volume informative, suggestive, and stimulating.

P. O. BELL