SUBSERIES OF A CONVERGENT SERIES

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In a recent paper J. D. Hill[1] has discussed the mean-value of the subseries of any absolutely convergent series $s=\sum u_n$. Simplifying his method by use of the Rademacher functions, we obtain a mapping of the subseries into the interval $0 \leqq x \leqq 1$ by defining[2]

$$\phi(x) = \sum_{n=1}^{\infty} \frac{1 + R_n(x)}{2} u_n. \tag{1}$$

Hill's result states that if $\sum |u_n|$ converges, then the mean-value is given by

$$\int_0^1 \phi(x)dx = s/2. \tag{2}$$

In the theorem below we point out the weakest condition on the series $\sum u_n$ for which this result persists.

LEMMA. *If* (1) *converges on a set of positive measure it converges almost everywhere.*

Let D be the set of points on which (1) converges. Let $x = a_1a_2a_3 \cdots$ (in binary notation) be a point of D. If a finite number of the a_i are changed then the new point still belongs to D, for by the definition of the Rademacher functions this operation changes only a finite number of the terms of the series (1). Then D is a "homogeneous" set not of measure 0; hence it must be of measure 1.[3]

THEOREM. *A necessary and sufficient condition that the series* (1) *converge on a set of positive measure is that the two series* $\sum u_n$ *and* $\sum u_n^2$ *converge. Then* (1) *converges almost everywhere and* (2) *is valid.*

(i) Suppose that (1) converges on a set of positive measure. Then it must, by the lemma, converge almost everywhere. Then there exist

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[1] Bull. Amer. Math. Soc. vol. 48 (1942) p. 103.

[2] The mapping is not 1-1 at the points $x=k/2^n$, but this does not affect the results. For the properties of the Rademacher functions used in this paper see Kacmarcz and Steinhaus, *Le système orthogonale de M. Rademacher*, Studia Mathematica vol. 2 (1939) p. 231.

[3] C. Visser, *The law of nought-or-one in the theory of probability*, Studia Mathematica vol. 7 (1938) pp. 146–147.

two points x_0 and $1-x_0$, symmetric in $x=1/2$, at which (1) converges. Inasmuch as $R_n(x_0)+R_n(1-x_0)=0$ we have from (1) that $\phi(x_0)+\phi(1-x_0)=u_n/2$, so that $\sum u_n$ converges. Hence $\sum u_n R_n(x)$ converges a.e., so that $\sum u_n^2$ converges.[4]

(ii) Suppose now that $s=\sum u_n$ and $\sum u_n^2$ converge. Then the series of (1) converges a.e. to a function $\phi(x)$. The $R_n(x)$ are orthonormal, so that by the Riesz-Fischer theorem the series $\sum u_n R_n(x)/2$ converges in the mean to a function of L^2; this function must coincide a.e. with $\phi(x)-s/2$.

To establish (2) note that by the Schwarz inequality

$$\begin{aligned}\int_0^1 \left| \phi(x) - s/2 - \sum_1^N (u_n/2)R_n(x) \right| dx \\ \leqq \left(\int_0^1 \left| \phi(x) - s/2 - \sum_1^N (u_n/2)R_n(x) \right|^2 dx \right)^{1/2} \\ = o(1) \qquad\qquad (N \to \infty).\end{aligned}$$

Since $\int_0^1 R_n(x)dx=0$, (2) is an immediate consequence.

We conclude with some remarks.

(i) Hill points out that if in our theorem we take $\sum u_n$ to be conditionally convergent then D is of the first category[1] though of measure 1.[5]

(ii) Ulam notes that if $\sum |u_n|$ converges then the set of values taken on by $\phi(x)$ is a perfect set and asks what perfect sets can be obtained this way.[5]

(iii) We can obtain part of the above theorem from the laws of 0 or 1 of probability.[6] For (1) may be regarded as a series of independent random variables of mean-value and standard deviation $u_n/2$.

(iv) The above method furnishes a simple proof of a theorem of Steinhaus: the series (1) is $(C, 1)$ summable a.e. if and only if $\sum u_n$ is $(C, 1)$ summable and $\sum u_n^2$ converges.[7]

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[4] Kacmarcz and Steinhaus, op. cit. p. 234.

[5] Written communication to the author.

[6] See, for example, Khintchine and Kolmogoroff, *Über Konvergenz von Reihen, deren Glieder durch den Zufall bestimmt werden*, Rec. Math. (Mat. Sbornik) N. S. vol. 32 (1925) p. 668.

[7] Incorporated into a paper of Paley and Zygmund, *On some series of functions* (2), Proc. Cambridge Philos. Soc. vol. 26 (1930) p. 473.