

86. W. K. Feller: *On a probability limit theorem of H. Cramér.*

Let  $\{X_k\}$  be mutually independent random variables whose first moments vanish and whose second moments are  $\sigma_k^2$ ; let  $s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$ . In various applications one is concerned with the distribution function  $Pr\{X_1 + \cdots + X_n > xs_n\}$ , where  $x \rightarrow \infty$  as  $n \rightarrow \infty$ . The simplest binomial case has been studied by A. Khintchine, P. Lévy, Smirnov and others. H. Cramér found a complete description of the asymptotic behavior of the above sums in the case where all  $X_k$  have the same distribution function. This theorem is generalized to the case of unequal components. The theorem is to serve as a base for a solution of the "problem of the iterated logarithm" in the general case. (Received November 21, 1942.)

87. W. K. Feller: *On the general form of the so-called law of the iterated logarithm.*

Let  $\{X_k\}$  be a sequence of mutually independent random variables whose first moments vanish and whose second moments are  $\sigma_k^2$ ; let  $s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$ . A sequence of numbers  $\phi_k \uparrow \infty$  is said to be of upper (lower) class if the probability that  $X_1 + \cdots + X_n > s_n \phi_n$  for infinitely many  $k$  is one (zero); any sequence  $\{\phi_k\}$  is either of upper or of lower class. A n.a.s. condition is found for a sequence  $\{\phi_k\}$  to belong to the upper class. It generalizes the condition found by Kolmogoroff and Erdős in the special case where  $X_k$  assumes the values  $\pm 1$  only, each with probability  $1/2$ ; however, it is different in form. The new theorem also contains a result of Marcinkiewicz and Zygmund on the necessity of the condition imposed by Kolmogoroff on the  $X_k$  in his proof of the law of the iterated logarithm. (Received November 21, 1942.)

88. P. G. Hoel: *On indices of dispersion.*

The sampling distribution of the index of dispersion for binomial and Poisson distributions is investigated by means of semi-invariants. Approximations to terms of order  $N^{-3}$  are obtained for the descriptive moments of the distribution, by means of which the accuracy of the chi-square approximation can be determined. (Received October 30, 1942.)

### TOPOLOGY

89. R. F. Arens: *Homeomorphism groups of a space.* Preliminary report.

Let  $A$  be a locally bicomact, locally connected Hausdorff space, and let  $G$  be a group of homeomorphisms of  $A$ . Then there is a certain topology  $N$  making  $G$  into a topological group (Pontrjagin, *Topological groups*, Princeton, 1939). This topology  $N$  is the weakest admissible topology that can be introduced into  $G$ , in this sense: Sets in  $G$  open by  $N$  are open by any other admissible topology  $M$ . A topology  $M$  for a group of homeomorphisms,  $G$ , of a space  $A$  is called admissible if by using that topology for  $G$  the two functions  $g(a)$  and  $g^{-1}(a)$ , where  $g \in G$  and  $a \in A$ , become continuous functions of both arguments  $g$  and  $a$  simultaneously. The topology  $N$  is determined by the following system of neighborhoods of the identity in  $G$ : Select in  $A$  an open set  $W$  whose closure is bicomact, and another open set whose closure  $K$  is contained in  $W$ . Then the set  $U$  of all  $g \in G$  which transform  $K$  into  $W$  is defined to be a neighborhood of the identity. The set of all such  $U$  together with all their finite

intersections form a complete system of neighborhoods for the identity homeomorphism in  $G$ . (Received November 16, 1942.)

90. G. D. Birkhoff: *Measure-preserving transformations of a planar ring without planar periodic points.*

Let there be given a planar ring  $R: a \leq r \leq b$  and a one-to-one continuous measure-preserving (that is, conservative) direct transformation  $T$  of  $R$  into itself which has the same rotation number  $\alpha$  along the circular boundaries  $C_a$  and  $C_b$ , so that it is not possible to infer the existence of periodic point groups by the aid of Poincaré's last geometric theorem. The note discusses the conclusions which may be drawn if in fact there are no periodic points whatsoever, in the special case  $T = RU$  ( $R^2 = U^2 = \text{identity}$ ); this special hypothesis as to the form of  $T$  is satisfied in the transformation  $T$  associated with the restricted problem of three bodies. Among other things, it is proved that there will then exist a qualitative integral  $\Theta(P) = \text{constant}$  such that  $\Theta[T(P)] = \Theta(P) + \alpha$  where  $\Theta$  is continuous in the polar coordinates  $(r, \theta)$  of  $P$  and increases by  $2\pi$  when  $P$  makes a positive circuit of the ring  $R$ . (Received December 1, 1942.)

91. D. G. Bourgin: *On a theorem of Goldstine's.*

A new, topological proof is given for Goldstine's theorem (Duke Math. J. vol. 4 (1938), pp. 125-131) relating  $w(E)$  completeness and reflexivity in a Banach space. (Received November 24, 1942.)

92. D. G. Bourgin: *Quasi norms in linear topological spaces.*

It is proved that a continuous quasi norm can always be defined on a locally bounded linear topological space. This answers a question of Hyers (Revista de Ciencias vol. 4 (1939) pp. 558-574) in the affirmative. The multiplier of the quasi norm is not uniquely determined by the l.t.s. and it is shown by means of an example that there need be no "best" multiplier. (Received November 24, 1942.)

93. D. E. Christie: *Net homotopy for compacta.*

Let  $R$  be a compactum embedded in a parallelotope  $P$ . Two mappings  $f$  and  $g$  of a topological space  $S$  into  $R$  are said to be neighborhood-homotopic if they are homotopic in every neighborhood of  $R$  in  $P$ . The notion of neighborhood-homotopy leads to corresponding homotopy groups. These groups are independent of the particular embedding of  $R$  in  $P$  since they are equivalent to similar groups defined in terms of the net (sense of Lefschetz) of all open coverings of  $R$ . This equivalence is demonstrated by means of a study of the Kuratowski mappings of neighborhoods of  $R$  into their nerves. The point of view adopted for homotopies is next applied to mappings. Both net and neighborhood definitions are developed: the former lead to easier proofs and wider applications, but the latter are more intuitive. The homotopy groups obtained from these new considerations are significant even for non-arcwise-connected spaces; they are suitably related to Čech homology even for spaces subject to no restrictions about local connectedness; yet they agree with the Hurewicz homotopy groups for absolute neighborhood retracts. (Received November 23, 1942.)

94. H. S. M. Coxeter: *The map-coloring of unorientable surfaces.*

Heawood proved that every map on an unbounded surface of characteristic  $K < 2$

can be colored with  $[F_K]$  colors, where  $F_K = (1/2)(7 + (49 - 24K)^{1/2})$ , and he conjectured that this number of colors is necessary as well as sufficient. Franklin disproved this conjecture in the case of the unorientable surface with  $K = 0$ . On the other hand, the necessity of  $[F_K]$  colors has been verified in many other cases by considering a map of this number of regions, each touching all the others; for example, Bose discovered a "balanced incomplete block design" which can be interpreted as a map of ten enneagons ( $K = -5$ ). In the present paper this map is shown to be related to two of the stellated icosahedra in ordinary space; and a map of nine octagons ( $K = -3$ ) is derived from a four-dimensional polytope. (Received November 5, 1942.)

95. Samuel Eilenberg and Saunders MacLane: *On a group construction by Hopf.*

In studying the influence of the fundamental group of a polyhedron on its second homology group  $H$ , Hopf (Comment. Math. Helv. vol. 14 (1942) pp. 257-309) constructs for each discrete group  $G$  an abelian group  $G_1^*$  and shows that when  $G$  is the fundamental group,  $G_1^*$  is a suitable quotient group of the second homology group. Hopf's definition of  $G_1^*$  uses a representation of  $G$  by means of generators and relations. In order to make the definition intrinsic the authors consider the central extensions of the group  $P$  of reals reduced mod 1 by the group  $G$ . The equivalence classes of such extensions form a compact abelian group  $\text{ExtCent} \{P, G\}$  whose character group is shown to be naturally isomorphic with  $G_1^*$ . The result is based on a generalization of a previous result of the authors concerning abelian group extensions (see Bull. Amer. Math. Soc. abstract 48-1-94). (Received November 23, 1942.)

96. L. M. Graves: *Metrisation of weak convergence in Banach spaces.*

In the topology of weak convergence in a Banach space, derived sets need not be closed, so that weak convergence cannot in general be expressed in terms of a metric. In this paper a class of cases is characterized in which a simple explicit solution can be given for the problem of the metrisation of weak convergence in certain subsets of a Banach space or of its adjoint. (Received October 29, 1942.)

97. Karl Menger and S. G. Reed: *On a surface not intersecting the set  $R_3^1$ .*

By the sum theorem of dimension theory  $R_3^1$ , the set of all points of the 3-space with exactly one rational coordinate, is 0-dimensional. As O. Schreier remarked,  $R_3^1$  intersects each surface  $z = f(x, y)$  where  $f$  is a continuous function defined on a square in the  $x, y$ -plane. The boundary of a neighborhood not intersecting  $R_3^1$  can be constructed by topping a cube with steppypyramids and iterating this process on each of the cubes of which the step consists. (Received October 28, 1942.)

98. A. N. Milgram: *A topologically invariant metric property of simple closed curves.*

Recently L. M. Blumenthal raised the question as to whether each simple closed curve contains three points which are vertices of an equilateral triangle. This question seems heretofore to have been unanswered even for closed plane curves. In this paper it is shown that: 1. Each simple closed plane curve contains the vertices of a triangle similar to any prescribed triangle. 2. Each closed polygon in euclidean  $n$ -space has the same property. (Received October 28, 1942.)

99. C. E. Rickart: *Decomposition of additive set functions.*

An additive set function  $x(e)$  defined in a  $\sigma$ -field  $\mathfrak{M}$  to a Banach space is said to satisfy property (A) provided  $\lim_{n \rightarrow \infty} x(e_n) = 0$ , for any disjoint sequence  $\{e_n\} \subset \mathfrak{M}$ . If  $x(e)$  satisfies (A) and  $\Gamma$  is any  $\sigma$ -ideal contained in  $\mathfrak{M}$ , then there exists a unique decomposition of  $x(e)$  of the form  $x(e) = x_1(e) + x_2(e)$ , where there exists  $e_0 \in \Gamma$  such that  $x_1(e) = x_1(ee_0)$  and  $x_2(e) = 0$  for every  $e \in \Gamma$ . A consequence of this result is that, for an arbitrary outer measure  $u(e)$  defined over  $\mathfrak{M}$ , every completely additive function  $x(e)$  can be decomposed uniquely into the form  $x(e) = x_1(e) + x_2(e)$ , where  $x_1(e) = x_1(ee_0)$ ,  $u(e_0) = 0$ , and  $x_2(e)$  is absolutely continuous relative to  $u(e)$  (that is,  $\lim_{n \rightarrow \infty} u(e_n) = 0$  implies  $\lim_{n \rightarrow \infty} x_2(e_n) = 0$ ). (Received November 19, 1942.)

100. R. L. Swain: *Approximate isometries in bounded spaces.*

Approximate isometries have been studied by S. M. Ulam and D. H. Hyers (abstract 47-9-427). The present paper exhibits an example to show that results of the type which they obtained cannot generally be obtained in bounded spaces. Also the following theorem is proved: Let  $M$  be a subset of a compact metric space  $S$  and let  $\eta$  be a positive number. Then there exists a positive number  $\epsilon$  such that if  $T$  is any  $\epsilon$ -isometry of  $M$  (within  $S$ ), there is an isometry  $U$  of  $M$  such that for each point  $x$  of  $M$ , the distance between  $T(x)$  and  $U(x)$  is less than  $\eta$ . (Received October 30, 1942.)

101. A. D. Wallace: *On non-alternating transformations.*

The  $H$ -spaces  $M$  and  $N$  are assumed to be compact and connected and  $\Delta$  will denote a continuous transformation from  $M$  onto  $N$ . If  $\Delta$  is non-alternating it is known that (a) for each  $p$ -chain  $Y$  there exists a unique  $p$ -chain  $X$  such that  $Y \subset \Delta X$ ; (b) if  $X$  is a chain then so also is  $\Delta X$ . To this may be added, if  $M$  is locally connected (c) if  $A$  and  $B$  are intersecting chains then  $\Delta(AB) = (\Delta A)(\Delta B)$ . This is no longer true if the hypothesis of local connectivity is deleted. It may be shown ( $\Delta$  non-alternating) that (d) if  $a \sim b$  and a continuum meets both  $\Delta^{-1}a$  and  $\Delta^{-1}b$  then the continuum contains points  $a_0, b_0$  with  $a = \Delta a_0, b = \Delta b_0$  such that  $a_0 \sim b_0$ ; (e) if  $K$  is a continuum and  $\Delta^{-1}K = A + B$  is a separation then there are points  $a_1, b_1$  in  $A$  and  $B$  such that  $a_1 \sim b_1$ . If it is assumed that  $\Delta$  satisfies (b), (d) and (e) then (a) holds even if  $\Delta$  alternates. The proof of (c) may be based on the following characterization of a chain: In order that the closed set  $C$  be a chain it is n.a.s. that it satisfy both (i) if  $a, b$  are distinct points of  $C$  and  $a \sim x \sim b$  then  $x$  is in  $C$  and (ii) if  $p$  separates the points  $s$  and  $t$  of  $C$  in the space, then  $p$  is in  $C$ . (Received October 26, 1942.)

102. G. S. Young: *Sets of axioms for the plane.* Preliminary report.

R. L. Moore has given several sets of axioms characterizing the plane, of which the Axioms 0-8 of his *Foundations of point set theory* (Amer. Math. Soc. Colloquium Publications vol. 13, 1932) are probably the most satisfactory. The purpose of this paper is to present several other sets of axioms also defining the plane, which postulate such properties as the two-sidedness of arcs, the non-existence of primitive skew curves, separation by open curves, non-separation by arcs, and so on. Axioms 0-2 of *Foundations* have been assumed throughout. The axiom sets considered divide into two classes: those in which the primary intent has been to state axioms equivalent to Moore's Axioms 3-5; and those in which the purpose was to furnish substitutes for Axioms 5-8. It is thought that these alternative approaches to the plane may prove useful in, for example, imbedding theorems. (Received October 28, 1942.)