PROJECTIONS OF THE PRIME-POWER ABELIAN GROUP OF ORDER p^m AND TYPE $(m-1, 1)$

ROSS A. BEAUMONT

1. **Introduction.** A function f of the subgroups of the group G upon the subgroups of the group H is called a projectivity of G upon H ($f(G)=H$) if the following hold.

(1) For every subgroup S of G, $f(S)$ is a subgroup of H.

(2) If S' is a subgroup of H, then there exists a subgroup S of G such that $f(S)=S'$.

(3) If S and T are subgroups of G, $S\leqq T$ is a necessary and sufficient condition that $f(S)\leqq f(T)$.

The correspondence f is a (1-1) correspondence which preserves the partial ordering of the set of subgroups of the group G.

Further, a projectivity f is called index-preserving if $[T:S] = [f(T):f(S)]$ for subgroups S of cyclic subgroups T of G; and f is called strictly index-preserving if $[T:S]=[f(T):f(S)]$ for subgroups S of subgroups T of G.

If G is the direct product of cyclic groups of order p, p a prime number, R. Baer[1] has given necessary and sufficient conditions that a group H be a projection of G. In particular he has shown that if the projectivity of G upon H is index-preserving, then G and H are isomorphic. Thus in a study of the projections of the prime-power abelian group of order p^m and type $(m-1, 1)$, we need consider only the case $m>2$.

Rottlaender[2] investigated the case $m=3$ and found necessary and sufficient conditions for the existence of a strictly index-preserving projectivity of the prime-power abelian group G of order p^3 and type $(2, 1)$ upon a group H.

In this note, Baer's general results are used to find the necessary and sufficient conditions for the existence of a projectivity of the prime-power abelian group G of order p^m and type $(m-1, 1)$ upon a group H.

2. **The necessary conditions.** If G is an abelian group of the type under consideration, $G=\{u_1\}\times\{u_2\}$ where u_1 is of order p^{m-1}, $m>2$,

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[1] R. Baer, *The significance of the system of subgroups for the structure of the group*, American Journal of Mathematics, vol. 61 (1939), pp. 1–44. Hereafter this paper will be referred to as B.

[2] Ada Rottlaender, *Nachweis der Existenz nicht-isomorpher Gruppen von gleicher Situation der Untergruppen*, Mathematische Zeitschrift, vol. 28 (1928), pp. 641–653.

and u_2 is of order p. If $f(G)=H$, it follows from known results[3] that f is index-preserving, and hence $H=\{f(\{u_1\}), f(\{u_2\})\}$ where $f(\{u_1\}) = \{u_1'\}$ is a cyclic group of order p^{m-1}, $f(\{u_2\})=\{u_2'\}$ is a cyclic group of order p. Thus $H=\{u_1', u_2'\}$ where u_1' and u_2' are independent generators of order p^{m-1} and p, respectively.

If K is any group, then K^p is the set of all pth powers of the elements of K, and K^p is a characteristic subset of K.

LEMMA 1. *$f(G^p)=H^p$ is a characteristic subgroup of H.*

PROOF. $G^p=\{u_1^p\}$ and $f(G^p)=f(\{u_1^p\})=f(\{u_1\})^p=\{u_1'^p\}$ since f is index-preserving. $\{u_1'^p\}\leqq H^p$. If x is in H^p, $x=y^p$ where y is in H. $f^{-1}(\{x\})=f^{-1}(\{y^p\})=f^{-1}(\{y\})^p=\{u_1^k u_2^n\}^p=\{u_1^{kp}\}$. $f(\{u_1^{kp}\})=f(\{u_1\})^{kp} = \{u_1'^p\}=\{x\}$. Hence $x=u_1'^{ikp}=(u_1'^p)^{ik}$ and x is in $\{u_1'^p\}$. Thus $H^p\leqq\{u_1'^p\}$ so that we have shown that $f(G^p)=\{u_1'^p\}=H^p$. Since H^p is a subgroup of H, it is a characteristic subgroup, and in particular normal.

Since $f(G)=H$ and $f(G^p)=H^p$, the index-preserving projectivity f of G upon H induces an index-preserving projectivity of G/G^p upon H/H^p. G/G^p is the direct product of two cyclic groups of order p and hence by the result[4] mentioned above, G/G^p and H/H^p are isomorphic. Thus H/H^p is abelian and H^p contains the commutator subgroup of H. This implies

$$u_2'^{-1}u_1'u_2' = u_1'^{1+jp}. \tag{1}$$

The following multiplication rules for elements in H follow from (1)

$$u_2'^{-1}u_1'^{h}u_2' = u_1'^{h(1+jp)}, \tag{2}$$

$$u_1'^{h}u_2'^{i} = u_2'^{i}u_1'^{h(1+jp)^i}, \tag{3}$$

$$\begin{aligned}(u_2'^{i}u_1'^{h})^{k} &= u_2'^{ki}u_1'^{h[1+(1+jp)^i+\cdots+(1+jp)^{(k-1)i}]}\\ &= u_2'^{ki}u_1'^{h[(1+jp)^{ki}-1]/[(1+jp)^i-1]}.\end{aligned} \tag{4}$$

It follows from (3) that the order of H is p^m so that we have proved the following theorem.

THEOREM 1. *If G is the prime-power abelian group of order p^m, $m>2$, and type $(m-1, 1)$ and if f is a projectivity of G upon a group H, then H is a prime-power group of order p^m generated by independent generators u_1' of order p^{m-1} and u_2' of order p such that $u_2'^{-1}u_1'u_2'=u_1'^{1+jp}$.*

If $p=2$, we derive the additional necessary condition.

[3] [B, Corollary 11.3].
[4] [B, Corollary 8.2].

LEMMA 2. *$j\equiv 0 \bmod 2$ so that $u_2'^{-1}u_1'u_2' = u_1'^{1+4w}$.*

PROOF. Since $f(\{u_1\}) = \{u_1'\}$, it follows from Baer's results[5] that there exists one and only one element u' in H such that $f(\{u_2\}) = \{u'\}$ and $f(\{u_2u_1\}) = \{u'u_1'\}$. Since $\{u'\} = \{u_2'\}$, $u' = u_2'^{\rho}$ where ρ is odd. $f(\{u_2u_1\}^2) = f(\{u_2u_1\})^2$ since f is index-preserving and we have

$$\begin{aligned} f(\{u_2u_1\}^2) &= f(\{u_1^2\}) = f(\{u_1\})^2 = \{u_1'^2\}, \\ f(\{u_2u_1\})^2 &= \{(u'u_1')^2\} = \{(u_2'^{\rho}u_1')^2\} \\ &= \{u_2'^{2\rho}u_1'^{[(1+2j)^{2\rho}-1]/[(1+2j)^{\rho}-1]}\} = \{u_1'^{(1+2)j^{\rho}+1}\}. \end{aligned}$$

Thus $u_1'^{(1+2j)^{\rho}+1} = u_1'^{2\gamma}$ where γ is odd and we have

$$\begin{aligned} (1+2j)^{\rho} + 1 &\equiv 2\gamma \bmod 2^{m-1}, \\ 2 + 2\rho j + [\rho(\rho-1)/2](2j)^2 + \cdots &\equiv 2\gamma \bmod 2^{m-1}, \\ 1 + \rho j + [\rho(\rho-1)/2]2j^2 + \cdots &\equiv \gamma \bmod 2^{m-2} \end{aligned}$$

and recall that $m>2$. Since γ is odd, the left member of the congruence is odd which implies ρj is even. Since ρ is odd, j is even, which completes the proof of the lemma.

3. Construction of a projectivity for groups satisfying the necessary conditions. If p is an odd prime and H is a group satisfying the necessary conditions, then H is either abelian or H is the unique non-abelian group $\{U_1, U_2\}$ where U_1 and U_2 are subject to the sole defining relations[6]

$$U_1^{p^{m-1}} = U_2^{p} = 1, \qquad U_2^{-1}U_1U_2 = U_1^{1+p^{m-2}}. \tag{5}$$

If H is abelian, then G and H are isomorphic and this isomorphism induces a projectivity of G upon H.

If H is non-abelian we will construct a projectivity of G upon H by establishing a correspondence between the cyclic subgroups and then extending this correspondence to a projectivity.

From (5) we find the following multiplication rule for elements of H

$$(U_2^{i}U_1^{h})^{k} = U_2^{ik}U_1^{h[k+ik(k-1)p^{m-2}/2]}. \tag{6}$$

Every element of G has the unique form $u_2^s u_1^r$ where $0 \leqq s < p$, $0 \leqq r < p^{m-1}$. If $s>0$, there exists an integer x, uniquely determined

[5] [B, (9.2), (d)].

[6] Carmichael, *Introduction to the Theory of Groups of Finite Order*, p. 132.

mod p^{m-1}, such that $sx\equiv 1 \bmod p^{m-1}$, $(x, p)=1$. Then $u_1^r=(u_1^r)^{sx}=u_1^{rx}=u_1^{ts}$ where $rx=t$. $u_2^s u_1^r=u_2^s u_1^{ts}$ and $\{u_2^s u_1^r\}=\{(u_2u_1^t)^s\}=\{u_2u_1^t\}$ where t is uniquely determined mod p^{m-1}. We define the following correspondence of the cyclic subgroups of G upon those of H

$$f(\{u_2u_1^t\}) = \{U_2U_1^t\}, \tag{7}$$

$$f(\{u_1^r\}) = \{U_1^r\}. \tag{8}$$

The correspondence f is a (1-1) correspondence of the set of cyclic subgroups of G upon a subset of the set of cyclic subgroups of H. To show that f is a (1-1) correspondence on the whole set of cyclic subgroups of H, it is only necessary to show that every cyclic subgroup of H has the form $\{U_2U_1^t\}$ or $\{U_1^r\}$ where t or r is uniquely determined mod p^{m-1}, respectively.

Every element of H has the unique form $U_2^sU_1^r$ where $0\leqq s<p$, $0\leqq r<p^{m-1}$. If $s>0$, there exists an integer x, uniquely determined mod p^{m-1}, such that $sx\equiv 1 \bmod p^{m-1}$, $(x, p)=1$. Then

$$\begin{aligned}\{U_2^sU_1^r\} &= \{(U_2^sU_1^r)^x\}\\ &= \{U_2^{sx}U_1^{r[x+(sx(x-1)/2)p^{m-2}]}\}\\ &= \{U_2U_1^{r[x+(sx(x-1)/2)p^{m-2}]}\}\end{aligned}$$

where $r[x+(sx(x-1)/2)p^{m-2}]$ is uniquely determined mod p^{m-1}.

The correspondence f preserves the indices of the cyclic subgroups since

$$f(\{u_2u_1^t\}^p) = f(\{u_1^{tp}\}) = \{U_1^{tp}\}$$

by (8),

$$f(\{u_2u_1^t\})^p = \{U_2U_1^t\}^p = \{U_1^{t[p+(p(p-1)/2)p^{m-2}]}\} = \{U_1^{tp}\},$$

by (7) and (6).

Since the only non-cyclic subgroups of H are those of the form $\{U_1^{p^\lambda}, U_2\}$, $0\leqq\lambda<m-1$, by extending f so that

$$f(\{u_1^{p^\lambda}, u_2\}) = \{U_1^{p^\lambda}, U_2\} \tag{9}$$

f becomes an index-preserving projectivity of G upon H.

Combining the above results with Theorem 1 we have this theorem.

THEOREM 2. *If G is the prime-power abelian group of odd order p^m, $m>2$, and type $(m-1, 1)$ then there exists a projectivity f of G upon a group H if, and only if, either H is isomorphic to G or H is the non-*

abelian group $\{U_1, U_2\}$ *where* U_1 *and* U_2 *are subject to the sole defining relations* (5).

If $p=2$ it follows from Lemma 2 that if $m=3$, then H is abelian and hence G and H are isomorphic groups. If $m>3$, it follows from Lemma 2 and from known results[7] that H is either abelian or the non-abelian group $\{U_1, U_2\}$ where U_1 and U_2 are subject to the sole defining relations

$$U_1^{2^{m-1}} = U_2^2 = 1, \qquad U_2U_1U_2 = U_1^{1+2^{m-2}} . \tag{10}$$

Since Baer has shown[8] that there exists an index-preserving projectivity of G upon this non-abelian group we have the following theorem.

THEOREM 3. *If* G *is the prime-power abelian group of order* 2^m, $m>2$, *and type* $(m-1, 1)$ *then*

(a) *if* $m=3$, *there exists a projectivity* f *of* G *upon a group* H *if and only if* G *and* H *are isomorphic groups;*

(b) *if* $m>3$, *there exists a projectivity* f *of* G *upon a group* H *if, and only if, either* G *and* H *are isomorphic groups or* H *is the non-abelian group* $\{U_1, U_2\}$ *where* U_1 *and* U_2 *are subject to the sole defining relations* (10).

UNIVERSITY OF WASHINGTON

[7] Carmichael, loc. cit., p. 133.

[8] [B, p. 11].