A NOTE ON HILBERT'S OPERATOR

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The transformation

$$(1)\ \mathfrak{H}f = \frac{1}{\pi} P\,V \int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt = \frac{1}{\pi} \lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} \frac{dt}{t} \{f(x+t) - f(x-t)\}$$

is well known to have the following properties:

LEMMA 1.[1] *When $1<p<\infty$, then $\mathfrak{H}f$ is a continuous (bounded) linear transformation with both domain and range $L_p(-\infty, \infty)$, and $\mathfrak{H}^2 f = -f$.*

LEMMA 2.[2] *When $f(t)\in L_1(-\infty, \infty)$, then $\mathfrak{H}f$ exists for almost all x in $(-\infty, \infty)$, but does not necessarily belong to $L_1(a, b)$, where a, b are arbitrary numbers $(-\infty \leqq a < b \leqq \infty)$; however $(1+x^2)^{-1}|\mathfrak{H}f|^q \in L_1(-\infty, \infty)$ when $0<q<1$. When f and $\mathfrak{H}f$ belong to $L_1(-\infty, \infty)$, then $\mathfrak{H}^2 f = -f$.*

The case $p=1$ appears to present the greatest difficulties. In the present note I shall deal with the set of elements $f(t)\in L_1(-\infty, \infty)$ for which $\mathfrak{H}f \in L_1(-\infty, \infty)$. In consequence of the lemmas, in this set or in $L_p(-\infty, \infty)$ $(1<p<\infty)$, $\mathfrak{H}f$ has no characteristic values other than $\pm i$. We shall start from the sets of characteristic functions and, incidentally, from the class $\mathfrak{H}_p$, the theory of which has been developed by E. Hille and J. D. Tamarkin; $\mathfrak{H}_p$ is the set of functions $F(z)$ $(z=x+iy)$ which, for $y>0$, are regular and satisfy the inequality

$$(2)\qquad \int_{-\infty}^{\infty} |F(x+iy)|^p dx \leqq M^p \quad \text{or} \quad |F(z)| \leqq M$$

for $0<p<\infty$ or $p=\infty$, respectively, where M depends on F and p only.[3] By $\mathfrak{K}_p$ we denote the corresponding class defined for $y<0$, and by $F(t)$, $G(t)$ the limit-functions[3] $(y\to 0;\ x=t)$ of elements $F(z)\in\mathfrak{H}_p$, $G(z)\in\mathfrak{K}_p$. By $\mathfrak{H}_p'$ and $\mathfrak{K}_p'$, respectively, we denote the two sets of those limit-functions, and by $\mathfrak{H}_p' \dotplus \mathfrak{K}_p'$ the smallest linear manifold

Received by the editors August 5, 1941.

[1] M. Riesz, Mathematische Zeitschrift, vol. 27 (1928), pp. 218–244.

[2] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937, §5.14. E. Hille and J. D. Tamarkin, Fundamenta Mathematicae, vol. 25 (1935), pp. 329–352. Comparing our notation with that of Hille-Tamarkin, we have $\mathfrak{H}f = -\tilde{f}$.

[3] Loc. cit., $1\leqq p<\infty$. T. Kawata, Japanese Journal of Mathematics, vol. 13 (1936), pp. 421–430, $0<p<\infty$. The limit-functions exist for almost all t in $(-\infty, \infty)$ and belong to $L_p(-\infty, \infty)$.

containing both $\mathfrak{H}_p'$ and $\mathfrak{K}_p'$. Obviously an element $f(t)$ belongs to $\mathfrak{H}_p' \dotplus \mathfrak{K}_p'$ if and only if it can be represented in the form

$$(3) \qquad f(t) = F(t) + G(t) = F(t) + \overline{F}_1(t), \quad F \in \mathfrak{H}_p', F_1 \in \mathfrak{H}_p', G \in \mathfrak{K}_p',$$

and this representation is unique, except for a constant when $p = \infty$.

Theorem 1 (b), as yet unpublished, is due to H. R. Pitt, Aberdeen, to whom I am greatly indebted.

We obtain the following results:

LEMMA 3. *Let $1 \leqq p \leqq \infty$; let the norm of an element $\phi(z)$ belonging to $\mathfrak{H}_p$ or $\mathfrak{K}_p$ be defined by*

$$(4) \qquad |\phi(t)|_p = \left\{ \int_{-\infty}^{\infty} |\phi(t)|^p dt \right\}^{1/p} \quad or \quad |\phi(t)|_p = \operatorname*{ess.u.b.}_{-\infty < t < \infty} |\phi(t)|$$

for $1 \leqq p < \infty$ or $p = \infty$, respectively. Then $\mathfrak{H}_p$ and $\mathfrak{K}_p$ are complete normed linear spaces, that is to say, (B) spaces in the terminology of Banach.

THEOREM 1. *Let $f(t) \in L_1(-\infty, \infty)$.* (a) *A necessary condition that $\mathfrak{H}f \in L_1(-\infty, \infty)$ is*

$$(5) \qquad \int_{-\infty}^{\infty} f(t)dt = 0.$$

(b) (*Pitt's theorem.*) *The condition is not sufficient.*

THEOREM 2. (a) *A necessary and sufficient condition that both f and $\mathfrak{H}f$ belong to $L_1(-\infty, \infty)$ is that f belongs to $\mathfrak{H}_1' \dotplus \mathfrak{K}_1'$.* (b) *With domain $\mathfrak{H}_1' \dotplus \mathfrak{K}_1'$, $\mathfrak{H}f$ is a linear closed[4] unbounded transformation in $L_1(-\infty, \infty)$.*

THEOREM 3. *The set $\mathfrak{H}_1' \dotplus \mathfrak{K}_1'$ is a non-closed subspace of L_1 and is nowhere dense in L_1. Its closure is the subset of L_1 satisfying* (5).

We note that, by Lemma 1 and by the argument which will be employed in the proof of Theorem 2(a), $\mathfrak{H}_p' \dotplus \mathfrak{K}_p' = L_p$ for $1 < p < \infty$.

We shall now give the proofs of the above results; some examples will be given at the end of this paper.

Proof of Lemma 3. We need only show that the space $\mathfrak{H}_p$ is complete. Let $\{F_n(z)\} \in \mathfrak{H}_p$, $n = 1, 2, \cdots$, be a sequence satisfying the condition of convergence $|F_n(t) - F_m(t)|_p \to 0$ $(m > n \to \infty)$. Then there exists an element $F(t) \in L_p$ such that $|F(t) - F_n(t)|_p \to 0$ as $n \to \infty$; we have to show that $F(t)$ is the limit-function of an element $\varphi(z) \in \mathfrak{H}_p$.

[4] That is to say, $f_n \in \mathfrak{H}_1' \dotplus \mathfrak{K}_1'$, $g_n = \mathfrak{H}f_n$ $(n = 1, 2, \cdots)$, $|f - f_n|_1 \to 0$ and $|g - g_n|_1 \to 0$ imply $g = \mathfrak{H}f$.

By a result due to Hille and Tamarkin,[5] $F_n(z)$ is represented by its "proper Poisson integral"

$$(6)\qquad F_n(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yF_n(t)dt}{(t-x)^2+y^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yF_n(t+x)}{t^2+y^2}\,dt, \qquad z = x+iy;\ y\to 0.$$

Let

$$(7)\qquad \varphi(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yF(t+x)}{t^2+y^2}\,dt.$$

For $y \geqq \epsilon > 0$, by Hölder's theorem, we have uniformly

$$(8)\qquad |\varphi(z) - F_n(z)| \leqq \pi^{-1}\epsilon^{-1/p}\,|(t^2+1)^{-1}|_{p'}\,|F(t)-F_n(t)|_p \to 0$$

as $n\to\infty$, where $1/p+1/p'=1$. Hence $\varphi(z)$ is a regular function for $y>0$, and it is obviously bounded when $p=\infty$.

Now let $1\leqq p<\infty$. By a well known convexity theorem,

$$(9)\qquad \begin{aligned} \int_{-\infty}^{\infty} |\varphi(x+iy)|^p dx &\leqq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{ydt}{t^2+y^2}\int_{-\infty}^{\infty} |F(t+x)|^p dx \\ &= \int_{-\infty}^{\infty} |F(x)|^p dx. \end{aligned}$$

Thus $\varphi(z)\in\mathfrak{H}_p$. By the same argument and by Fatou's theorem,

$$\begin{aligned} |\varphi(t) - F_n(t)|_p &\leqq \liminf_{y\to 0} \left\{ \int_{-\infty}^{\infty} |\varphi(z) - F_n(z)|^p dx \right\}^{1/p} \\ &\leqq |F(t) - F_n(t)|_p \to 0. \end{aligned}$$

By (8), the result holds for $p=\infty$. Therefore $F(t)\equiv\varphi(t)$, which completes the proof.

To prove Theorem 2(a) we require a result which we deduce from theorems by Hille and Tamarkin:

Lemma 4. *A necessary and sufficient condition that $f(t)$ belongs to $L_p(-\infty, \infty)$ $(1\leqq p<\infty)$ and that $\mathfrak{H}f=if$ or $\mathfrak{H}f=-if$ is that $f(t)$ belongs to $\mathfrak{H}_p'$ or $\mathfrak{K}_p'$, respectively.*

Let $f(t)\in\mathfrak{H}_p'$ and $f(t)=\varphi(t)+i\psi(t)$. Since $f(t)$ is the limit-function $(y\to 0,\ x=t)$ of an element $F(z)\in\mathfrak{H}_p$, and since $F(z)$ is represented by its proper Poisson integral, we have[6] $\psi(x) = -\mathfrak{H}\varphi$ and, by Lemmas 1

[5] Loc. cit., Theorem 2.1 (ii), $1\leqq p<\infty$. The result holds for $p=\infty$.

[6] Hille and Tamarkin, loc. cit., Theorem 3.1. For $p=2$, the lemma is an easy consequence of Theorem 95, Titchmarsh, loc. cit.

and 2, $\varphi(x)=\mathfrak{H}\psi$; therefore $\mathfrak{H}f=\mathfrak{H}(\varphi+i\psi)=-\psi+i\varphi=if$. Conversely, let $f(t)\in L_p$ and $\mathfrak{H}f=if$. For $y>0$, the function

$$F(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}\,dt$$

is regular and representable by its proper Poisson integral, and its limit function is $(1/2)\{f(x)-i\mathfrak{H}f\}=(1/2)(f(x)+f(x))=f(x)$.[6] Hence $F(z)\in\mathfrak{H}_p$, and so $f(t)\in\mathfrak{H}_p'$, which proves the lemma.

Proof of Theorem 2(a). Let $f\in L_1$ and $\mathfrak{H}f\in L_1$, then the functions

$$F=(1/2)(f-i\mathfrak{H}f),\qquad G=(1/2)(f+i\mathfrak{H}f)$$

belong to L_1; by Lemma 2, $\mathfrak{H}^2f=-f$, and so $\mathfrak{H}F=iF$, $\mathfrak{H}G=-iG$. By Lemma 4, we have $F\in\mathfrak{H}_1'$, $G\in\mathfrak{K}_1'$, and so $f=F+G\in\mathfrak{H}_1'\dot{+}\mathfrak{K}_1'$.

Conversely, let $f\in\mathfrak{H}_1'\dot{+}\mathfrak{K}_1'$. Then, by (3) with $p=1$, and by Lemma 4,

$$\mathfrak{H}f=\mathfrak{H}F+\mathfrak{H}G=i(F-G)\in L_1(-\infty,\infty),$$

which proves Theorem 2(a). Part (b) will be proved after Theorem 3.

To prove Theorem 1, we need a further result due to Hille and Tamarkin.[7]

Lemma 5. *Let $1\leqq p<\infty$, Let $\phi(t)$ belong to L_p and possess a Fourier transform $\psi(x)$,*

$$\psi(x)=(2\pi)^{-1/2}\ \underset{N\to\infty}{\text{l.i.m.}}\ \text{index } p'\int_{-N}^{N}\phi(t)e^{-itx}dt,\qquad 1/p+1/p'=1.$$

Then $\phi(t)\in\mathfrak{H}_p'$ or $\mathfrak{K}_p'$ if and only if $\psi(x)$ vanishes in $(-\infty,0)$ or in $(0,\infty)$, respectively.

For completeness we add the following result:

Lemma 5′. *Let $2<p<\infty$ and let $\phi(t)$ belong to L_p and have no Fourier transform in $L_{p'}$. Then $\phi(t)\in\mathfrak{H}_p'$ or $\mathfrak{K}_p'$ if and only if there is a sequence $\{\phi_n(t)\}$ belonging to $\mathfrak{H}_p'$ or $\mathfrak{K}_p'$ and satisfying the hypotheses of Lemma* 5 *and such that $|\phi(t)-\phi_n(t)|_p\to 0$ as $n\to\infty$.*[8]

[7] Loc. cit., Lemma 4.2, and Annals of Mathematics, (2), vol. 34 (1933), pp. 606–614, Theorem 3.

[8] The proof is similar to that given for the generalization of a theorem due to Paley-Wiener; H. Kober, Quarterly Journal of Mathematics, vol. 11 (1940), pp. 66–80, Theorem 2(b). Let $\phi(t)\in\mathfrak{H}_p'$ and $|\phi(t)-f_n(t)|_p\to 0$, where $f_n(t)$ $(n=1,2,\cdots)$ has a Fourier transform in $L_{p'}$; then the functions $\phi_n(t)=(1/2)(f_n-i\mathfrak{H}f_n)$ have the desired properties. The converse is proved by Lemma 3.

By Theorem 2 and Lemma 5, $f(t)$ can be represented as the sum of two functions $F(t)$, $G(t)$ belonging to L_1 and such that their Fourier transforms $\varphi(x)$, $\psi(x)$ vanish for $x<0$ or $x>0$, respectively. By continuity, they also vanish at $x=0$; so does the Fourier transform of $f(t)$, which gives (5). For the Fourier transform of an element $F(t)\in L_1$ is continuous in $(-\infty, \infty)$.

To prove Theorem 1(b), take $f_1(t)=t^{-1}\log^{-2} t$ and $f_2=2/\log 2$ in $(0, 1/2)$, $f_1(t)=f_2(t)=0$ otherwise. Let $f(t)=f_1(t)-f_2(t)$, then obviously $f(t)$ belongs to $L_1(-\infty, \infty)$ and satisfies (5). But $\mathfrak{H}f$ does not belong to $L_1(-\infty, \infty)$, since

$$\int_{-1/2}^{0} | \mathfrak{H}f_1 | \, dx = \infty,$$

$$\int_{-1/2}^{0} | \mathfrak{H}f_2 | \, dx = \frac{2}{\pi \log 2}\int_{-1/2}^{0} \left| \log \left| 1 - \frac{1}{2x} \right| \right| dx < \infty .$$

For, in $(0, 1/2)$, we have

$$\mathfrak{H}[f_1; -x] = \frac{1}{\pi}\int_0^{1/2} \frac{\log^{-2} t \, dt}{(x+t)t} \geqq \frac{1}{\pi}\int_0^{x} \frac{\log^{-2} t \, dt}{(x+t)t}$$
$$\geqq \frac{1}{\pi}\int_0^{x} \frac{\log^{-2} t \, dt}{2xt} = \frac{1}{2\pi x | \log x |};$$

hence $\mathfrak{H}f_1$ does not belong to $L_1(-1/2, 0)$, which proves the theorem.

Proof of Theorem 3. Let E be the subset of L_1 satisfying (5). By Theorems 1 and 2, $\mathfrak{H}_1' \dotplus \mathfrak{K}_1'$ is a subset of E and different from E. It is easy to see that E is closed in L_1. We are left to show that E is the closure of $\mathfrak{H}_1' \dotplus \mathfrak{K}_1'$.

Let $f(t)$ be a step-function belonging to E. Denoting by $\mathfrak{e}(t)$ the step-function which is equal to 1 in $(0, 1)$ and to zero otherwise, we can represent $f(t)$ by a finite sum $\sum a_n \mathfrak{e}(t/b_n)$ ($b_n \lessgtr 0$, a_n complex). By (5), $\sum a_n | b_n | =0$, and so

$$\pi\mathfrak{H}f = \sum a_n \left(\log \left| 1 - \frac{b_n}{x} \right| \right) \operatorname{sgn} b_n = O(x^{-2}), \qquad x \to \pm \infty.$$

Hence $\mathfrak{H}f\in L_1$, $f\in \mathfrak{H}_1' \dotplus \mathfrak{K}_1'$. We can now approximate to any $f(t)\in E$ by a sequence $\{f_n(t)\}$ $(n=1, 2, \cdots)$ of step-functions belonging to $\mathfrak{H}_1' \dotplus \mathfrak{K}_1'$. Let $f(t)$ satisfy (5), and let $\{g_n(t)\}$ be a sequence of step-functions such that $|f(t)-g_n(t)|_1 \to 0$ as $n\to\infty$. Take

$$f_n(t) = g_n(t) - \mathfrak{e}(t)\int_{-\infty}^{\infty} g_n(\xi)d\xi, \qquad n = 1, 2, \cdots .$$

Then $f_n(t)$ is a step-function, $f_n(t)$ satisfies (5); therefore $f_n \in \mathfrak{H}_1' \dotplus \mathfrak{K}_1'$. Finally, by (5), we have

$$|f - f_n|_1 = \left| f - g_n - \mathfrak{e}\int_{-\infty}^{\infty} \{f(\xi) - g_n(\xi)\} d\xi \right|_1$$
$$\leqq |f - g_n|_1 + \left| \int_{-\infty}^{\infty} \{f(\xi) - g_n(\xi)\} d\xi \right| \leqq 2|f - g_n|_1,$$

which tends to zero as $n \to \infty$. Thus E is the closure of $\mathfrak{H}_1' \dotplus \mathfrak{K}_1'$.

The set E is nowhere dense in L_1. For when $f \in L_1$, then, given $\epsilon > 0$, any element $g(t) = f(t) + \delta\mathfrak{e}(t)$ $(0 < |\delta| < \epsilon)$ belongs to the sphere $|g - f|_1 < \epsilon$, while g does not belong to E; when f belongs to L_1 but not to E, then no element g of the sphere $|g - f|_1 < |\int f(t)dt|$ belongs to E. Thus we have proved the theorem.

Proof of Theorem 2(b). In the domain $\mathfrak{H}_1' \dotplus \mathfrak{K}_1'$, by Lemma 2, we have $i\mathfrak{H}(i\mathfrak{H}f) = f$; hence $i\mathfrak{H}f$ is involutory. By Lemma 3, both $\mathfrak{H}_1'$ and $\mathfrak{K}_1'$ are closed spaces in L_1. Therefore $i\mathfrak{H}f$, and therefore $\mathfrak{H}f$, is closed; for a linear involutory transformation in a (B) space is closed if and only if the spaces of the characteristic functions are closed.[9] By Theorem 3, $\mathfrak{H}_1' \dotplus \mathfrak{K}_1'$ is not closed. Therefore $\mathfrak{H}f$ is not bounded in this domain; for a linear closed transformation in a (B) space is continuous if and only if its domain is closed.[10] Thus we have proved the theorem.

The following are examples for the case $f \in L_1(-\infty, \infty)$, $\mathfrak{H}f \in L_1(-\infty, \infty)$. We may start from Lemma 4,[11] but it is easier to make use of Theorem 2.

(1) Let $T_1(z)$ or $T_2(z)$ be polynomials of degree $\alpha > 0$ or $\beta > 0$ and such that they have no zeros for $y \geqq 0$ or $y \leqq 0$, respectively; let a, b be any numbers such that $-\infty < a < -1/\alpha$, $-\infty < b < -1/\beta$. Then, on a suitable Riemann surface, any branch of $\{T_1(z)\}^a$ $(y > 0)$ or $\{T_2(z)\}^b$ $(y < 0)$ belongs to $\mathfrak{H}_1$ or $\mathfrak{K}_1$, respectively. When $f(t) = \{T_1(t)\}^a + \{T_2(t)\}^b$, by Lemma 4, we have $\mathfrak{H}f = i\{T_1(x)\}^a - i\{T_2(x)\}^b \in L_1$.

(2) Let $\varphi_1(z) = (1 - \cos\alpha z)z^{-2}$, $\varphi_2(z) = \{\sin\alpha z - 2\sin(\alpha z/2)\}z^{-2}$, $\alpha > 0$, and let $f(t) = A\varphi_1(t)e^{i\alpha t} + B\varphi_2(t)e^{-i\alpha t}$; then $\mathfrak{H}f = iA\varphi_1(x)e^{i\alpha x} - iB\varphi_2(x)^{-i\alpha x}$, and $f \in L_1$, $\mathfrak{H}f \in L_1$. It can be shown that this result holds when $\varphi_j(z)$ $(j = 1, 2)$ are integral functions such that $\varphi_j(t) \in L_1$ and that, for any $\epsilon > 0$, $|\varphi_j(z)| < K_\epsilon \exp\{(\alpha + \epsilon)|z|\}$; in this way we can construct

[9] H. Kober, Proceedings of the London Mathematical Society, (2), vol. 44 (1938), pp. 453–465, Theorem 6′(a).

[10] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 41, Theorem 7. Probably the converse is well known.

[11] Or from Theorem 3.1, Hille and Tamarkin, loc. cit.

all integral functions $f(z)$ satisfying the conditions $f(t)\in L_1$, $\mathfrak{H}f\in L_1$, $|f(z)|<K_{f,\epsilon}\exp\{(2\alpha+\epsilon)|z|\}$. The proof is based upon a result due to Plancherel and Pólya.[12]

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[12] Commentarii Mathematici Helvetici, vol. 10 (1937–1938), pp. 110–163, §27.

THE BEHAVIOR OF CERTAIN STIELTJES CONTINUED FRACTIONS NEAR THE SINGULAR LINE

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1. **Introduction.** We consider here continued fractions of the form[1]

$$(1.1)\qquad f(z)=\frac{g_0}{1}+\frac{g_1z}{1}+\frac{(1-g_1)g_2z}{1}+\frac{(1-g_2)g_3z}{1}+\cdots,$$

in which $g_0\geqq 0$, $0\leqq g_n\leqq 1$, $(n=1, 2, 3, \cdots)$, it being agreed that the continued fraction shall terminate in case some partial numerator vanishes identically. There exists a monotone non-decreasing function $\phi(u)$, $0\leqq u\leqq 1$, such that

$$(1.2)\qquad f(z)=\int_0^1\frac{d\phi(u)}{1+zu};$$

and, conversely, every integral of this form is representable by such a continued fraction. Put $M(f)=\text{l.u.b.}_{|z|<1}|f(z)|$. Then $M(f)\leqq 1$ if and only if the continued fraction can be written in the form

$$(1.3)\qquad f(z)=\frac{h_1}{1}+\frac{(1-h_1)h_2z}{1}+\frac{(1-h_2)h_3z}{1}+\cdots,$$

in which $0\leqq h_n\leqq 1$, $(n=1, 2, 3, \cdots)$. These functions are analytic in the interior of the z-plane cut along the real axis from $z=-1$ to $z=-\infty$.

The principal object of this paper is to prove the following theorem:

THEOREM 1.1. *If $0<h_n<1$, $(n=1, 2, 3, \cdots)$, and $h_n\rightarrow 1/2$ in such a way that the series $\sum|h_n-1/2|$ converges, then the function $f(z)$ given*

Presented to the Society, October 25, 1941; received by the editors August 14, 1941.

[1] H. S. Wall, *Continued fractions and totally monotone sequences*, Transactions of this Society, vol. 48 (1940), pp. 165–184.