NORMAL VARIETIES AND BIRATIONAL CORRESPONDENCES

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1. **Introduction.** As one advances into the general theory of algebraic varieties, one reluctantly but inevitably reaches the conclusion that there does not exist a general theory of birational correspondences. This may sound too reckless a statement or too harsh a criticism, especially if one thinks of the fundamental role which birational transformations are supposed to have in algebraic geometry. Nevertheless our conclusion is in exact agreement with the facts and it is made with constructive rather than with critical intentions. It is true that the geometers have a fairly good intuitive idea of what happens or what may happen to an algebraic variety when it undergoes a birational transformation; but the only thing they know with any certainty is what happens in a thousand and one special cases. All these special cases—and they include all Cremona transformations—are essentially reducible to one special but very important case, namely, the case in which the varieties under consideration are nonsingular(that is, free from singular points). One can give many reasons for regarding as inadequate any theory which has been developed exclusively for nonsingular varieties. One rather obvious reason is that we have as yet no proof that every variety of dimension greater than 3 can be transformed birationally into a nonsingular variety.[1] But there are other, less transient, reasons. Were such a proof available, it would still be advisable to develop the theory of algebraic varieties, *as far as possible*, without restricting oneself to nonsingular projective models. This certainly would be the correct program of work from an arithmetic standpoint. I have a distinct impression that my friends the algebraists have not much use anyway for the resolution of the singularities. All they want is a general uniformization theorem, and now that they have it, they are content.

The following consideration will perhaps carry greater weight with the geometers. It turns out, as I have found out at some cost to myself, that we have to know a lot more about birational correspondences than we know at present before we can even attempt to carry

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[1] The resolution of the singularities of three-dimensional varieties will be carried out in a forthcoming paper of mine.

out the resolution of the singularities of higher varieties. A general theory of birational correspondences is a necessary prerequisite for such an attempt. I shall have occasion later on to indicate some difficult questions concerning birational correspondences which arise in connection with the resolution of singularities.

2. **Birational correspondences and valuations.** From a formal point of view, there is nothing mysterious about a birational transformation. If $V=V_r^n$ is an irreducible r-dimensional algebraic variety in an n-dimensional projective space, the coordinates $\xi_1, \xi_2, \cdots, \xi_n$ of its general point are algebraic functions of r independent variables, and the field $\Sigma=K(\xi_1, \xi_2, \cdots, \xi_n)$ generated by these functions is *the field of rational functions on* V. Here K denotes the field of constants (in the classical case K is the field of complex numbers). If $V'=V_r'^m$ is another irreducible algebraic variety, with general point $(\xi_1', \xi_2', \cdots, \xi_m')$ and associated field $\Sigma'=K(\xi_1', \xi_2', \cdots, \xi_m')$, then the two varieties V and V' are *birationally equivalent* if the two fields Σ and Σ' are simply isomorphic: $\Sigma/K \cong \Sigma'/K$. A birational transformation is merely the process of passing from one variety to another, birationally equivalent, variety.

The difficulties begin when we wish to associate with this purely formal process a *geometric transformation*, that is, a correspondence between the points of the two varieties. From the equations of the transformation, in which the ξ''s are given as rational functions of the ξ's and vice versa, it is not difficult to conclude that the transformation sets up a (1, 1) correspondence between the non-special points of V and the non-special points of V'. The points of either variety for which the equations of the transformation fail to define corresponding points on the other variety are referred to as special points in the sense that they lie on certain algebraic subvarieties, of dimension less than r. In the classical case, considerations of continuity allow us to complete the definition of the correspondence also for these special points. In the abstract case we use valuation theory instead, as follows:

A *valuation* of the field Σ is an homomorphic mapping v of the multiplicative group $\Sigma-0$ (that is, the element zero excluded) upon an ordered additive abelian group Γ, which satisfies the well known valuation axioms: (1) $v(\omega_1\cdot\omega_2)=v(\omega_1)+v(\omega_2)$; (2) $v(\omega_1\pm\omega_2)\geqq\min\{v(\omega_1), v(\omega_2)\}$; (3) $v(\omega)\neq 0$, for some ω in Σ; (4) $v(c)=0$, for all constants $c\neq 0$. We put $v(0)=+\infty$.

In the case of algebraic functions of one variable, every valuation arises from a branch of our curve V. The value $v(\omega)$ is then *the order* *of* ω at the branch and is an integer. Positive and negative $v(\omega)$ signify, respectively, that the center of the branch is a zero or a pole of the function ω, while if $v(\omega)=0$ then the function-theoretic value of ω at the center of the branch is a finite constant, different from zero. In this special case, it is clear that from an algebraic standpoint the function-theoretic values of the elements of the field are the cosets of the *valuation ring* $\mathfrak{B}$ $\{\omega\in\mathfrak{B}\leftrightarrow v(\omega)\geqq 0\}$, with respect to its subset consisting of the elements ω such that $v(\omega)>0$. This subset is a prime divisorless ideal $\mathfrak{p}$, and so the cosets form indeed a field; the field of complex numbers, in the classical case. This consideration is independent of the dimension of the field and can therefore be applied directly to the general case. It is therefore always possible to associate with any valuation v of Σ a mapping f of the elements of Σ upon the elements of another field (and the symbol ∞), *the field of residual classes* of $\mathfrak{B}$ mod $\mathfrak{p}$, and we may speak of $f(\omega)$, $\omega\in\Sigma$, as being the *function-theoretic value* of ω (if $v(\omega)<0$, then $f(\omega)=\infty$). This field is the so-called *residue field* of the valuation. However, if $r>1$ then the residue field may be a transcendental extension of the ground field K. Its degree of transcendence s over K, or briefly, *its dimension*, is at most $r-1$, and is referred to as the dimension of the valuation. A zero-dimensional valuation is called *a place* of the field Σ. The function-theoretic values of the elements of Σ at a given place are *constants*, that is, either elements of K or algebraic quantities over K.

Given a valuation v and a projective model V of Σ, with general point $(\xi_1, \xi_2, \cdots, \xi_n)$, it is permissible to assume that the function-theoretic values of the ξ's are different from ∞, since we may subject the coordinates ξ_i to an arbitrary projective transformation. Then the polynomials in the ξ's which have function-theoretic value zero form a prime ideal in the ring of all polynomials in the ξ's. This prime ideal defines an irreducible algebraic subvariety W of V. This subvariety W we call *the center of the valuation* v on the variety V. The dimension of W cannot exceed the dimension of the valuation. In particular, *the center of a place is always a point of* V.

The following geometric picture of a valuation is suggestive, although not entirely adequate. A zero-dimensional valuation, that is, a place, with center at a point P, corresponds to a way of approaching P along some one-dimensional branch, which may be algebraic, analytic, or transcendental. Similarly an s-dimensional valuation with an s-dimensional center W corresponds to a way of approaching W along an $(s+1)$-dimensional branch through W.

After these preliminaries, we define the birational correspondence between two birationally equivalent varieties V and V' as follows:

DEFINITION 1. *Two subvarieties W and W' of V and V', respectively, correspond to each other if there exists a valuation of the field Σ whose center on V is W and whose center on V' is W'.*

Note that our definition does not treat points in any privileged fashion. Any subvariety of V is treated as an element, rather than as a set of points. This procedure is much more convenient than the usual one in which corresponding loci are defined as loci of corresponding points.

In the study of the birational correspondence between V and V', it is found convenient to introduce a third variety $\overline{V}$ which is birationally related to both V and V', the so-called *join of V and V'* (or *the variety of pairs of corresponding points* of V and V'). $\overline{V}$ is defined as follows. We adjoin to Σ a new transcendental η_0 and we regard the $n+1$ quantities $\eta_0, \eta_1=\eta_0\xi_1, \cdots, \eta_n=\eta_0\xi_n$ as the *homogeneous coordinates* of the general point of V. Similarly, the $m+1$ quantities $\eta_0'=\eta_0, \eta_1'=\eta_0\xi_1', \cdots, \eta_m'=\eta_0\xi_m'$ will be the homogeneous coordinates of the general point of V'. The $(n+1)(m+1)$ quantities $\omega_{ij}=\eta_i\eta_j'$ can be regarded as the homogeneous coordinates of the general point of a variety birationally equivalent to V and to V'. This variety is our $\overline{V}$, the join of V and V'. The birational correspondence between $\overline{V}$ and V has the property that to any subvariety of $\overline{V}$ there corresponds a unique subvariety of V; in particular, to every point of $\overline{V}$ there corresponds a unique point of V. Similarly, for $\overline{V}$ and V'. Thus, both V and V' are single-valued transforms of $\overline{V}$. The properties of the birational correspondence between V and V' can be readily derived from the properties of the birational correspondences between $\overline{V}$ and V and between $\overline{V}$ and V'. We therefore replace one of the two varieties V, V', say V', by the join $\overline{V}$, that is, from now on we shall always assume that *V is a single-valued transform of V'.*

3. Fundamental loci; geometric preliminaries. On the basis of Definition 1, it is easy to prove that the points of V to which there correspond more than one point on V' constitute an algebraic subvariety F of V, and that every subvariety W of V to which there correspond more than one subvariety W' on V' must lie on F. This variety F is called the *fundamental locus* of the birational correspondence, and every W which lies on F is a *fundamental* variety. This is not our final definition, but it will do for the moment. Note that the fundamental locus on V' is an empty set, in view of our assumption that V is a single-valued transform of V'.

What corresponds on V' to a fundamental variety W of V? To this question we have a complete answer in the case of a nonsingular

V. We have, namely, in this case the following two fundamental theorems (see van der Waerden, *Algebraische Korrespondenzen und rationale Abbildungen*, Mathematische Annalen, vol. 110 (1934)):

A. *If W is an irreducible s-dimensional fundamental variety of V, then the transform of W is an algebraic subvariety of V' whose irreducible components are all of dimension greater than s.*

B. *The transform of the fundamental locus F is a pure $(r-1)$-dimensional subvariety of V'.*

The following examples show that both theorems fail to hold for singular models.

(1) If P is a point of V at which V is *locally reducible*, that is, if in the neighborhood of P the variety V consists of ν $(\nu > 1)$ analytical r-dimensional branches, then in *special cases* it turns out that V is the projection of another variety V' on which these ν branches become separated.[2] Then the point P will be the projection of ν distinct points of V', and this contradicts Theorem A.

(2) Let Q be a ruled quadric surface in S_3 and let V be the three-dimensional cone which projects Q from a point O not in S_3. We take another copy of Q, say Q', which we now imagine as being immersed in an S_5. Let l be a line in S_5 which does not meet the S_3 containing the quadric. We set up a (1, 1) projective correspondence between the points P' of l and the lines p of one ruling of the quadric. Let V' be the irreducible three-dimensional variety generated by the planes (P', p), where P' and p are corresponding elements in the above projectivity. It is easy to set up a birational correspondence between V and V' in which to the planes (O, p) there correspond the planes (P', p). There will be no fundamental points on V', while O will be the only fundamental point on V. To the point O there corresponds on V' the line l, in contradiction with Theorem B.

One has the feeling that the second example does more damage than the first, because the first counterexample could be explained away on the basis that the point P, as origin of ν analytical branches, should not be regarded at all as a "point" of the variety, but rather as a point of the ambient projective space at which ν "points" of the variety accidentally happened to come together. This explanation, if stripped of all metaphysics, can have only one mathematical meaning, namely, it means, by implication, that in the general theory of birational correspondence, we should restrict ourselves to varieties

[2] Whether this is true generally, is not at all obvious and, in fact, has never been proved.

which are locally irreducible at each point. What interests us in this argument is the formal admission that some kind of restriction as to the type of varieties to be studied is necessary in the theory of birational correspondences. Whether or not the restriction to locally irreducible varieties is the right one, is a debatable matter. For one thing, it is not certain that in making such a restriction we are not being too stingy, unless we can prove that the branches of a variety can always be separated by the method of projection. This is probably true and should not be too difficult to prove. Actually we regard this restriction as being too generous. For, besides the requirement that the varieties V of our hypothetical restricted class satisfy Theorem A, we find it essential that these varieties also satisfy the following additional condition:

C. *If to a point P of V there corresponds a unique point P' of V', then the birational correspondence, regarded as an analytical transformation, is regular at P.*

The geometric meaning of this condition can be roughly indicated as follows. If this condition and condition A are satisfied for a given variety V, then the analytical structure of the neighborhood of any point P of V cannot be affected by a birational transformation, unless this transformation blows up P into a curve, or a surface and so on (always provided we replace the transform V' of V by the join $\overline{V}$; compare with §2). In particular, it is not possible to simplify any further the type of singularity which V possesses at P without doing a thing as radical as that of spreading out that singular point into a variety of dimension greater than 0. From this point of view, condition C can be looked upon as a sort of maximality condition.

In the case of algebraic curves it follows readily from this geometric interpretation that the only curves which satisfy conditions A and C are the nonsingular curves. But already in the case of surfaces we get a much wider variety of types. For instance, it can be proved that the surfaces in S_3 which satisfy our conditions are the surfaces which have only isolated singularities.

We now proceed to define arithmetically the varieties of our restricted class. We call these varieties *locally normal*. Included among these varieties are those which I have called *normal* varieties. The algebraic operation which plays a fundamental role in our arithmetic approach to the geometric questions just outlined is that of the *integral closure of a ring in its quotient field*. Theorem C follows in a relatively simple fashion from our arithmetic definitions and from some well known theorems in valuation theory. Theorem A lies much deeper and its proof is more difficult. As to Theorem B, it definitely must be sacrificed when we are dealing with singular varieties.

4. **Locally normal and normal varieties.** We use the homogeneous coordinates $\eta_0, \eta_1, \cdots, \eta_n$ of the general point of V (§2) and we define the *quotient ring* $Q(P)$ of any point P of V as the ring of all quotients $f(\eta)/g(\eta)$, where f and g are forms of like degree in $\eta_0, \eta_1, \cdots, \eta_n$ and where $g \neq 0$ at P. In other words, the quotient ring $Q(P)$ consists of all functions in our field Σ which have a definite and finite value at P. In a similar fashion, we define the quotient ring $Q(W)$ of any irreducible algebraic subvariety W of V by the condition that $g(\eta) \neq 0$ on W (that is, that g should not vanish at every point of W).

One is led to the consideration of quotient rings when one examines the equations of a birational correspondence between V and another variety V'. For it is seen immediately that if the nonhomogeneous coordinates $\xi_1', \cdots, \xi_m'$ of the general point of V' belong to the quotient ring $Q(W)$ of a given W on V, then to W there corresponds a unique subvariety W' on V' and moreover $Q(W')$ will be a subring of $Q(W)$. If $Q(W') = Q(W)$, then also W will be the only subvariety of V which corresponds to W'. In this case we say that the birational correspondence is *regular at* W, or *along* W. In particular, if the birational correspondence is regular at a point P of V, then as an analytical transformation it is regular in the neighborhood of P. Therefore, the quotient ring of a point determines uniquely the analytical structure of the neighborhood of the point. In the sequel we shall say that a birational correspondence is *regular* on V if it is regular at each point of V.

DEFINITION 2. *A variety V is locally normal along a subvariety W, if the quotient ring $Q(W)$ is integrally closed in its quotient field* (*that is, in* Σ).

DEFINITION 3. *V is locally normal, if it is locally normal at each point.*

The last definition refers only to points. The reason for this is the following: If V is locally normal at one point P of a subvariety W of V, then it is also locally normal along W. Hence if V is locally normal, in the sense of Definition 3, it is also locally normal along any W.

It is not difficult to show that V is locally normal if and only if the following condition is satisfied: *If $\mathfrak{C}$ is the conductor of the ring $K[\eta_0, \eta_1, \cdots, \eta_n]$ with respect to the integral closure of this ring in its quotient field, then the subvariety of V determined by the* (*homogeneous*) *ideal $\mathfrak{C}$ is empty.* This implies that either $\mathfrak{C}$ has no zeros at all, or its only zero is the trivial one: $(0, 0, \cdots, 0)$. Therefore, $\mathfrak{C}$ is either the unit ideal or is a primary ideal belonging to the irrelevant prime ideal $(\eta_0, \eta_1, \cdots, \eta_n)$.

If $\mathfrak{C}$ is the unit ideal, we say that V is *normal*, that is, we give the following definition:

DEFINITION 4. *A variety V is normal if the ring $K[\eta_0, \eta_1, \cdots, \eta_n]$ is integrally closed in its quotient field.*

It can be proved that the singular manifold of a locally normal r-dimensional variety is of dimension less than or equal to $r-2$ (in particular, *a locally normal curve is nonsingular*). The converse is not generally true, except in the case $r=1$, *since a nonsingular curve is always locally normal*. However, *the converse is true for hypersurfaces*, that is, for V_r's in an S_{r+1}, Thus any surface in S_3 is locally normal if and only if it has a finite number of singularities.

It is important to point out that *nonsingular varieties are always locally normal.*

Our definitions clearly indicate that normal varieties can differ from locally normal varieties only by some property at large, since locally they cannot be distinguished from each other. This difference at large is put into evidence by the following characterization of normal varieties, due to Muhly: *A variety V is normal if and only if the hypersurfaces of its ambient space, of any given order m, cut out on V a complete linear system.* Now it can be proved that locally normal varieties are characterized by the completeness of the above linear systems *for sufficiently high values of m.* This last result, in conjunction with the fact that every nonsingular V is locally normal, contains as a special case of the well known lemma of Castelnuovo concerning nonsingular curves. This lemma plays an important role in Severi's proof of Riemann-Roch's theorem for surfaces.

We have already pointed out that for locally normal varieties Theorems A and C hold true. Moreover it can be shown that these are the only varieties for which these theorems are true. It may be added that there is really no great loss of generality in confining the theory of birational correspondences to locally normal varieties. For it can be shown that any variety V determines uniquely, *to within regular birational transformations*, a locally normal variety V' which is birationally equivalent to V and which is such that: (a) to each point P' of V' there corresponds a unique point P of V, and we have always: $Q(P') \supseteq Q(P)$; (b) to any point P of V there corresponds a finite number of points on V'; (c) if V is locally normal at P, then the birational correspondence between V and V' is regular at P. It is not difficult to show that these properties of the birational correspondence between

V and V' imply that, to within a regular birational transformation, *V is the projection of V' from a center S_k which does not meet V'.*

It would be of interest to characterize the locally normal varieties for which Theorem B holds. It can be proved that the following condition is sufficient for the validity of Theorem B: If P is any point of V and if W is any $(r-1)$-dimensional subvariety through P, then a sufficiently high multiple of W should be *locally* (that is, at P) complete intersection of V with an hypersurface of the ambient space. In terms of ideal theory, this means that in the quotient ring of P a sufficiently high power of any minimal prime ideal should be *quasi-gleich* (in the sense of van der Waerden) to a principal ideal. This condition gives us a good insight into the "real" reason of the validity of Theorem B for nonsingular varieties; for we know that if P is a simple point, then every minimal prime ideal in $Q(P)$ is itself a principal ideal.

5. **Monoidal transformations.** I should now like to discuss briefly a special class of birational correspondences which seem to be very useful in the theory of singularities, whether we deal with the resolution of singularities or with the analysis of the composition of a singularity from the standpoint of infinitely near points. These special transformations are the hyperspace analogue of plane quadratic transformations, and they are therefore of importance also for the general theory of birational correspondences.

When we are dealing with locally normal varieties V, we find it most convenient to define fundamental varieties of a birational correspondence between V and another variety V', as follows:

DEFINITION 5. *An irreducible subvariety W of V is fundamental if a corresponding subvariety W' of V' exists such that $Q(W) \not\supseteq Q(W')$.*

We know from §4 that if $Q(W) \supseteq Q(W')$, then W' is the only subvariety which corresponds to W. Hence if W is fundamental, then the relation $Q(W) \not\supseteq Q(W')$ is true for any W' which corresponds to W. If the birational correspondence has no fundamental points on V and on V', then $Q(W) = Q(W')$, for *any* two corresponding subvarieties W and W', and the transformation is regular on V (and on V'). We do not regard two birational transformations as being essentially distinct if they differ only by a regular birational transformation.

We consider a birational correspondence between two locally normal varieties V and V', and we again restrict ourselves to the case in which the given correspondence has no fundamental points on V'. Then V' is in regular birational correspondence with the join $\overline{V}$ of V'

and V, and we may replace V' by $\overline{V}$. Consequently, we assume that the equations of the birational correspondence between V and V' are of the form:

$$\rho\eta_{ij} = \eta_i\phi_j, \qquad i = 0, 1, \cdots, n; j = 0, 1, \cdots, m,$$

where $\eta_0, \eta_1, \cdots, \eta_n$ are the homogeneous coordinates of the general point of V and where the ϕ_j are forms of like degree in the η's. Here ρ is a factor of proportionality and the η_{ij} are the homogeneous coordinates of the general point of V'. It can be shown that the fundamental locus on V is given by the base manifold of the *linear system* $\lambda_0\phi_0 + \cdots + \lambda_m\phi_m = 0$, provided we first drop all fixed $(r-1)$-dimensional components of the system. In terms of ideal theory, it means that we first write each principal ideal (ϕ_j) as a power product of minimal primes, say $(\phi_j) = \mathfrak{A}\mathfrak{B}_j$, where $\mathfrak{A}$ is the highest common divisor of $(\phi_0), \cdots, (\phi_m)$. Then the fundamental locus is given by the ideal $(\mathfrak{B}_0, \mathfrak{B}_1, \cdots, \mathfrak{B}_m)$. This ideal is of dimension less than or equal to $r-2$.

The special transformations which we wish to discuss are those for which $(\phi_0, \phi_1, \cdots, \phi_m)$ is itself a prime ideal, of dimension $s \leqq r-2$ or differs from a prime ideal by an irrelevant primary component. For the lack of a better name, we call them *monoidal transformations*.[3] The irreducible subvariety W of V defined by the ideal $(\phi_0, \phi_1, \cdots, \phi_m)$ is called the *center* of the transformation. It is not difficult to see that a change of the base of the ideal $(\phi_0, \cdots, \phi_m)$ does not essentially affect the transformation. A *quadratic transformation* is a special case of a monoidal transformation, the center is in that case a point.

The effect of a monoidal transformation consists in that the center W is spread out into an $(r-1)$-dimensional irreducible subvariety W' of V'. Moreover, *points of W which are simple both for V and W, correspond to simple points of W'*. This is the main reason why a monoidal transformation is a useful tool in the resolution of singularities, since while it may conceivably simplify some singular points which lie on its center, it does not introduce new singularities.[4]

There are two outstanding problems concerning monoidal trans-

[3] With some non-essential modifications, and without their projective trimmings, the space Cremona transformations, known as monoidal transformations, are monoidal transformations in our sense.

[4] There is one exception: to a simple point of V which is *singular* for W there may correspond singular points of V! For this reason it is usually advisable to "smooth out" W, that is, to resolve the singularities of W, before one applies the monoidal transformation.

formations which play a role in the problem of resolution of singularities, but which at the same time are decisively of interest in themselves. We proceed to outline these questions.

PROBLEM 1. *Given any birational correspondence between two nonsingular models V and V', and assuming that there are no fundamental points on V', show that the birational transformation can be decomposed into monoidal transformations.*

In other words, the question is to show that for nonsingular models the monoidal transformations form a set of generators of the birational group. It is very likely that this decomposition exists also when only V' is nonsingular.

PROBLEM 2. *Given any birational correspondence between two arbitrary (not necessarily nonsingular) models V and V', and assuming as before that there are no fundamental points on V', show that the fundamental varieties on V can be eliminated by monoidal transformations.*

By this I mean that it is asked to transform V by a sequence of monoidal transformations into another variety V^* such that the birational correspondence between V^* and V' has no fundamental points on V^*.

As to Problem 1, we have a proof in the case of surfaces. In this case, the result can be regarded as a generalization of the well known theorem of Noether on the decomposition of plane Cremona transformations into quadratic transformations, although Noether's theorem is not a special case of this general result. It may be well to clarify the connection between the two results. In the first place, a quadratic Cremona transformation is not at all a quadratic transformation in our sense. Our quadratic transformation has only one ordinary fundamental point, and its inverse has no fundamental points at all, while a plane quadratic transformation and its inverse both have three fundamental points, which in special cases may be infinitely near points. For this reason a plane Cremona transformation can never be a quadratic transformation in our sense. The transform of a plane π under a quadratic transformation in our sense is not a plane, but a certain rational surface M in S_5, or any other surface in regular birational correspondence with M. Of course, an ordinary quadratic transformation between two planes π and π' can be expressed as a product of quadratic transformations in our sense, or more precisely as the product of 3 successive quadratic transformations and of 3 inverses of quadratic transformations. Since our proof takes care of surfaces over abstract fields K, it yields immediately a corresponding result for the fundamental varieties W of dimension $r-2$ in the general case, for the adjunction of certain $r-2$ transcendentals to the ground field will make a surface out of the variety V and a point out of W. In particular, the decomposition into monoidal transformations is thus established for birational correspondences between nonsingular three-dimensional varieties, provided the correpondence has only fundamental curves, but no isolated fundamental points.

In applications of Problem 2, the main interest lies in the elimination of the *simple* fundamental varieties of V. In this case we have a complete proof, provided the resolution theorem is granted for varieties of *dimension two less* than the dimension of V. Thus, in the case of three-dimensional varieties we have to use only a thing as little as that of the resolution of singularities of an algebraic curve. It is clear that Problem 2 is to be viewed as a step in an inductive proof of the general theorem of the resolution of singularities rather than as a problem for the solution of which we first need that general theorem. The really important problem is Problem 1. Its solution seems to be essential for the resolution of singularities of higher varieties. Thus, it is possible to carry out the resolution of singularities of three-dimensional varieties, because we have the theorem of local uniformization for the varieties of this dimension, plus the solution of Problem 1 for surfaces over abstract fields of constants.

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