CONFORMAL GEOMETRY OF ONE-PARAMETER FAMILIES OF CURVES

ROBERT COLEMAN, JR.

A single regular analytic arc in the plane has no conformal differential invariants. The conformal theory of curvilinear angles was initiated by Kasner,[1] and has been elaborated by him and others. The present paper is concerned with conformal differential invariants of a real one-parameter family of regular analytic arcs in the plane. We assume that the family is defined in some region R of the (x, y)-plane by an equation of the form: $u(x, y)=\text{constant}$, where u is a single-valued function which satisfies the conditions: (1) u is analytic in the region R, (2) u assumes real values for real values of x and y, (3) $u_x^2+u_y^2$ does not vanish in R. By a conformal transformation we shall mean a real conformal transformation, nonsingular in R. Our principal results are: When a family $u=c$ is transformed conformally into a family $U=c$, the parameters of the two families being the same, the quantity $\Delta\equiv(u_{xx}+u_{yy})/(u_x^2+u_y^2)$, and certain conformally invariant derivatives of Δ are unaltered. There exist rational functions of Δ and these derivatives which are independent of the parameter in terms of which the family $u=\text{constant}$ is expressed. We obtain a geometric interpretation of the invariants and apply the results to a generalization of isothermal families.

1. **The invariants.** Let $U(X, Y)=c$ be a one-parameter family in the (X, Y)-plane. Let this plane be mapped conformally on the (x, y)-plane by the transformation

$$X = X(x, y), \qquad Y = Y(x, y), \tag{1.1}$$

where

$$X_x = Y_y, \quad X_y = -Y_x, \quad X_{xx} + X_{yy} = Y_{xx} + Y_{yy} \equiv 0. \tag{1.2}$$

The family $U(X, Y)=c$ is transformed into $u(x, y)=c$ where $u(x, y) \equiv U[X(x, y), Y(x, y)]$. By differentiating this last identity we obtain:

$$U_X X_x + U_Y Y_x = u_x, \qquad U_X X_y + U_Y Y_y = u_y. \tag{1.3}$$

These equations together with (1.2) give

$$u_x^2 + u_y^2 = J(U_X^2 + U_Y^2), \quad \text{where } J \equiv X_x^2 + X_y^2. \tag{1.4}$$

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[1] Proceedings of the International Congress at Cambridge, 1912.

Differentiating the first of (1.3) with respect to x and the second with respect to y, and making use of (1.2) we obtain

$$u_{xx} + u_{yy} = J(U_{XX} + U_{YY}). \tag{1.5}$$

This equation with (1.4) gives

$$\frac{u_{xx} + u_{yy}}{u_x^2 + u_y^2} = \frac{U_{XX} + U_{YY}}{U_X^2 + U_Y^2}. \tag{1.6}$$

Hence the quantity $\Delta \equiv (U_{XX} + U_{YY})/(U_X^2 + U_Y^2)$ is unaltered.

Let dS denote the linear element of the (X, Y)-plane, and ds that of the (x, y)-plane. Under the mapping (1.1) we have

$$dS^2 = J ds^2. \tag{1.7}$$

Let $Q(X, Y)$ be a differentiable function defined in a region R' of the (X, Y)-plane, C' any arc with continuous tangent in R'. Let dQ/dS denote the directional derivative of Q along C' at some point P'. Let C and P be the images of C' and P' under the mapping (1.1), and suppose that under the mapping we have $Q[X(x, y), Y(x, y)] \equiv q(x, y)$, then

$$\left(\frac{dQ}{dS}\right)_{P'} = J^{-1/2}\left(\frac{dq}{ds}\right)_P.$$

This equation together with (1.4) gives

$$\frac{1}{(U_X^2 + U_Y^2)^{1/2}} \frac{dQ}{dS} = \frac{1}{(u_x^2 + u_y^2)^{1/2}} \frac{dq}{ds}. \tag{1.8}$$

Equation (1.8) holds in particular when d/dS denotes differentiation along a curve of the family $U = c$. In what follows we give d/dS this meaning. The curve $U = c_0$ is so oriented that if θ denotes the angle from the positive direction of the X-axis to the positive direction of $U = c_0$, then

$$\cos\theta = U_Y[U_X^2 + U_Y^2]^{-1/2}, \qquad \sin\theta = -U_X[U_X^2 + U_Y^2]^{-1/2}.$$

Since angles are preserved by the mapping (1.1), equation (1.8) holds if the derivatives are taken along orthogonal trajectories of $U = c$ and $u = c$. The trajectories are so oriented that the angle from the positive direction of $U = c_0$ to the positive direction of its orthogonal trajectory is $\pi/2$. We shall use the symbol d/dN to denote differentiation along an orthogonal trajectory of $U = c$.

Suppose that $I(U_X, U_Y, U_{XX}, \cdots)$ is any differentiable function of the derivatives of the first n orders of $U(X, Y)$ which is unaltered by the mapping (1.1), that is,

$$I(U_X, U_Y, U_{XX}, \cdots) \equiv I(u_x, u_y, u_{xx}, \cdots).$$

Then from (1.8) we see that the quantities

$$\frac{1}{[U_X^2 + U_Y^2]^{1/2}} \frac{dI}{dS}, \qquad \frac{1}{[U_X^2 + U_Y^2]^{1/2}} \frac{dI}{dN}$$

will also be unaltered; furthermore, they depend only on the derivatives of U of the first $n+1$ orders. We have seen that Δ is unaltered. Hence from Δ we may obtain an infinite number of other invariant functions of the derivatives of U by repeated application of the operators

$$\frac{d}{d\lambda} \equiv \frac{1}{[U_X^2 + U_Y^2]^{1/2}} \frac{d}{dS}, \qquad \frac{d}{d\lambda_N} \equiv \frac{1}{[U_X^2 + U_Y^2]^{1/2}} \frac{d}{dN}.$$

If $Q(X, Y)$ is any p times differentiable function we will write

$$\frac{d^pQ}{d\lambda^p} \equiv Q_{(p\lambda)}, \qquad \frac{d^pQ}{d\lambda_N^p} \equiv Q_{(p\lambda_N)}.$$

From (1.4) and (1.7) we obtain $[U_X^2+U_Y^2]dS^2=(u_x^2+u_y^2)ds^2$. Suppose that P_1', P_2' are two points on the same curve $U=c_0$ and P_1, P_2 are the corresponding points on the curve $u=c_0$. Since the transformation is nonsingular, by hypothesis, we have

$$\int_{P_1'}^{P_2'} [U_X^2 + U_Y^2]^{1/2} dS = \int_{P_1}^{P_2} (u_x^2 + u_y^2)^{1/2} ds$$

where the integrals are taken along $U=c_0$ and $u=c_0$, respectively. We have this theorem:

THEOREM 1. *When a family $U=c$ is transformed by a conformal transformation into a family $u=c$, the quantities*

$$(1.9) \qquad \int_{P_1'}^{P_2'} (U_X^2 + U_Y^2)^{1/2} dS; \ \Delta, \ \Delta_{(p_1\lambda)(p_2\lambda_N)(p_3\lambda)\cdots}$$

are invariant.

It is to be noted that the quantities (1.9) are expressed in terms of a particular parameter, and are not independent of changes of parameter. For isothermal families, Δ and its derivatives in (1.9) vanish identically when the parameter is suitably chosen. For other families we have this theorem.

THEOREM 2. *For non-isothermal families, there exist rational functions of the quantities* (1.9) *which are independent of the parameter in terms of which the family is expressed.*

To establish this theorem we note first of all that since $U_X^2 + U_Y^2 \neq 0$ by hypothesis, X, Y may be expressed in terms of U, V where $V = k$ gives the orthogonal trajectories of $U = c$. We have

$$
(1.10)\quad
\begin{aligned}
\frac{\partial}{\partial U} &\equiv \left(X_U \frac{\partial}{\partial X} + Y_U \frac{\partial}{\partial Y} \right) \\
&= \frac{1}{U_X^2 + U_Y^2} \left[U_X \frac{\partial}{\partial X} + U_Y \frac{\partial}{\partial Y} \right] \equiv \frac{d}{d\lambda_N}, \\
-\mu \frac{\partial}{\partial V} &= -\mu \left(X_V \frac{\partial}{\partial X} + Y_V \frac{\partial}{\partial Y} \right) \\
&= \frac{1}{U_X^2 + U_Y^2} \left[U_Y \frac{\partial}{\partial X} - U_X \frac{\partial}{\partial Y} \right] \equiv \frac{d}{d\lambda},
\end{aligned}
$$

where

$$\mu \equiv \frac{V_Y}{U_X} = -\frac{V_X}{U_Y}.$$

Suppose now that the parameter of a non-isothermal family is changed by means of the equation $\overline{U} = h(U)$. Then if we denote by $\Delta(P')$ the value of Δ at a point P' we have

$$(1.11)\qquad \phi \overline{\Delta}(P') = \frac{d}{dU} (\log \phi) + \Delta(P'),$$

where $\phi = dh/dU$ and the bar refers to the parameter $\overline{U}$. With a point $P'(U, V)$ we may associate a second point P_0' on the same curve $U = c_0$ as follows. Let $V = k_0$ be some *fixed* orthogonal trajectory cutting every curve $U = c$ in the region R'; then P_0' is the point (U, k_0). From (1.1) we have

$$(1.12)\qquad \phi[\overline{\Delta}(P') - \overline{\Delta}(P_0')] = \Delta(P') - \Delta(P_0').$$

From (1.10) we obtain

$$(1.13)\qquad \phi \frac{d}{d\overline{\lambda}_N} \equiv \phi \frac{\partial}{\partial \overline{U}} = \frac{\partial}{\partial U} = \frac{d}{d\lambda_N},$$

$$(1.14)\qquad \phi \frac{d}{d\overline{\lambda}} \equiv -\overline{\mu}\phi \frac{\partial}{\partial V} = -\mu \frac{\partial}{\partial V} \equiv \frac{d}{d\lambda}.$$

Now let $L(P'; P_0')$ be any rational function of Δ and a finite number of the derivatives in (1.9). Let $\overline{L}(P'; P_0')$ denote the same function of the corresponding quantities for the parameter $\overline{U}$, and suppose that L and $\overline{L}$ satisfy

$$\phi^n \overline{L}(P'; P_0') = L(P'; P_0'), \tag{1.15}$$

then we can obtain two other such functions involving the derivatives in (1.9) up to the $(p+1)$th order if L involves them up to the pth order. For (1.14) and (1.15) give $\phi^{n+1}\overline{L}_{\overline{\lambda}} = L_\lambda$. Hence L_λ is one such function. Secondly, (1.13) and (1.15) give

$$\phi^{n+1}\overline{L}_{\overline{U}} + n\phi^n \overline{L} \frac{d}{dU} (\log \phi) = L_U.$$

This, by virtue of (1.11) and (1.15) is equivalent to

$$\phi^{n+1}\overline{M} = M,$$

where

$$M \equiv L_U + nL\Delta(P'), \qquad \overline{M} \equiv \overline{L}_{\overline{U}} + n\overline{L}\overline{\Delta}(P'),$$

so that M is a second such function. Now from (1.12), $\Delta(P') - \Delta(P_0')$ is one function of the type L above. Hence for each of the derivatives in (1.9) we can obtain a function rational in it and the lower derivatives which satisfies a relation of the type (1.15). The function ϕ in (1.15) can be eliminated as follows. We have seen that

$$\lambda(P') \equiv \int_{P_0'}^{P'} [U_X^2 + U_Y^2]^{1/2} dS,$$

where the integral is taken along a curve $U = c_0$, is invariant. Clearly

$$\overline{\lambda}(P') = \phi\lambda(P')$$

so that

$$\overline{\lambda}^n \overline{L} = \lambda^n L.$$

This completes the proof.

2. **Geometric interpretation.** We have seen that when a family $u(x, y) = c$ is transformed conformally into a family $U(X, Y) = c$, the quadratic form

$$(u_x^2 + u_y^2)(dx^2 + dy^2) \tag{2.1}$$

is invariant. We may associate with the family $u = c$ a surface Σ, whose linear element is (2.1). Σ undergoes an isometric transformation when the (x, y)-plane undergoes a conformal transformation. The surface Σ is in conformal correspondence with the (x, y)-plane, corresponding points having the same coordinates (x, y), so that the family $u(x, y)=c$ in the plane is the conformal image of a family $u(x, y)=c$ in Σ. Suppose that we have a family $U(X, Y)=c$ in the (X, Y)-plane, with associated surface Σ'. If there is an isometric mapping between Σ and Σ' carrying $u=c$ into $U=c$, then this mapping induces a conformal transformation between the two planes which carries the family $u=c$ into $U=c$. Hence we have the following theorem:

THEOREM 3. *A necessary and sufficient condition that two families be conformally equivalent is that they admit parameters such that their associated surfaces are isometric, with the images of the families in the surfaces corresponding.*

From (1.10) we obtain

$$dx^2 + dy^2 = \frac{1}{u_x^2 + u_y^2}(du^2 + \mu^{-2}dv^2). \tag{2.2}$$

Hence the linear element of Σ is

$$d\sigma^2 = du^2 + \mu^{-2}dv^2. \tag{2.3}$$

Using a standard formula for the geodetic curvature of a curve $u=c$ in Σ we obtain[2]

$$\frac{1}{\rho} = \left(\log\frac{1}{\mu}\right)_u. \tag{2.4}$$

By virtue of (1.10) this is equivalent to

$$\frac{1}{\rho} = \left(\log\frac{1}{\mu}\right)_u = \Delta. \tag{2.5}$$

As a consequence of this equation we have this theorem:

THEOREM 4. *The geodetic curvature of a curve $u=c_0$ through a point P of Σ is measured by the value of Δ at P. The operators $d/d\lambda$, $d/d\lambda_N$ denote differentiation with respect to the arc length of the curves $u=c$ and their orthogonal trajectories respectively.*

If K denotes the Gaussian curvature of Σ we have $K=-\mu[\mu^{-1}]_{uu}$. By virtue of (2.5) this is equivalent to

$$-K = \Delta^2 + \Delta_u. \tag{2.6}$$

[2] Eisenhart, *Differential Geometry*, 1909, p. 134.

THEOREM 5. *The Gaussian curvature of Σ and its various rates of change along the orthogonal net determined by $u=c$ is measured by rational functions of Δ and its derivatives in* (1.9).

If two families are conformally equivalent, then for a suitable choice of parameters, the equations $U(X, Y)=u(x, y)$, $V(X, Y)=v(x, y)$, where $V=K$, $v=k$ give the orthogonal trajectories, define a conformal correspondence. If the associated surface of $U=c$ has the linear element $dU^2+M^{-2}dV^2$, then by (2.3) we have that $M(U, V)\equiv\mu(U, V)$. Conversely, if $M(U, V)\equiv\mu(U, V)$, the families are equivalent by Theorem 3. The function μ is an integrating factor of $-u_y dx+u_x dy=0$.

THEOREM 6. *Let the families $u=c$, $v=k$ form an orthogonal net. Let $\overline{U}(X, Y)=C$, $\overline{V}(X, Y)=K$ form a second such net. If $\mu(u, v)$ is an integrating factor of $-u_y dx+u_x dy=0$, then a necessary and sufficient condition for the conformal equivalence of the families $u=c$, $\overline{U}=C$ is that there exist parameters $U=F(\overline{U})$, $V=G(\overline{V})$ such that $-U_Y dX+U_X dY=0$ admits an integrating factor of the form $\mu(U, V)$.*

The reciprocal relationship between a family and its associated surface is given by this next theorem:

THEOREM 7. *The orthogonal trajectories of a family $u=c$ are conformal images of a system of geodesics of the surface Σ associated with $u=c$. Conversely, if Σ is a real surface which admits conformal representation on the plane such that a family of its geodesics corresponds to the orthogonal trajectories of a family $u=c$, then the family $u=c$ can be so parameterized that the associated surface is isometric with Σ.*

The first part of the theorem is an immediate consequence of equation (2.3). Now suppose $u(x, y)=c$ is a family in the plane, with orthogonal trajectories $v(x, y)=k$. Let Σ be a real surface which can be mapped conformally onto the plane so that a system of its geodesics $\bar{v}=b$ go over into the family $v=k$. We may take on Σ a system of geodesic parameters consisting of $\bar{v}=b$ and their orthogonal trajectories, $\bar{u}=a$, and the linear element of Σ takes the form $d\bar{u}^2+\zeta(\bar{u}, \bar{v})d\bar{v}^2$. The conformal correspondence between Σ and the plane may be written $\bar{v}=f(v)$, $\bar{u}=g(u)$. But these equations define a change of parameters in the (x, y)-plane, so that if we change the parameters of the families $u=c$ and $v=k$ in accordance with these equations we have $dx^2+dy^2=\xi(\bar{u}, \bar{v})[d\bar{u}^2+\zeta(\bar{u}, \bar{v})d\bar{v}^2]$. Because of (2.2)

$$\xi(\bar{u}, \bar{v}) = \frac{1}{\bar{u}_x^2+\bar{u}_y^2}, \qquad \zeta(\bar{u}, \bar{v}) = [\mu(\bar{u}, \bar{v}]^{-2}.$$

Hence if we express the family $u=c$ in the form $\bar{u}=a$ we see from (2.3) that the linear element of the associated surface is given by $d\sigma^2=d\bar{u}^2+\zeta(\bar{u},\ \bar{v})d\bar{v}^2$. This completes the proof.

3. **Generalization of isothermal families.** Isothermal families are characterized by the fact that they admit parameters such that $\Delta\equiv 0$, so that when an isothermal family is referred to such a canonical parameter the associated surface is of constant zero curvature. We shall determine those families which admit parameters such that the associated surface is of constant curvature. Such families are of special character and will be referred to as *families of constant curvature.*

Given a family $u(x, y)=c$. Let the parameter be changed to $\bar{u}=h(u)$. Denote by K and $\bar{K}$ the curvatures of the corresponding associated surfaces. By use of equations (1.11), (1.13), and (2.6) we obtain

$$(3.1)\qquad [\phi(u)]^2\bar{K} = K - \Delta\frac{d}{du}[\log\phi(u)] - \frac{d^2}{du^2}[\log\phi(u)],\qquad \phi\equiv\frac{dh}{du}\cdot$$

It follows from this equation that when a family of constant curvature is referred to an arbitrary parameter, Δ satisfies an equation of the form

$$\Delta_u + \Delta^2 + \Delta\frac{d}{du}(\log\phi) + \frac{d^2}{du^2}(\log\phi) + a\phi^2 = 0$$

where a is a constant. This equation together with the fact that

$$\Delta\equiv\frac{u_{xx}+u_{yy}}{u_x^2+u_y^2},\qquad \frac{\partial}{\partial u}=\frac{1}{u_x^2+u_y^2}\left[u_x\frac{\partial}{\partial x}+u_y\frac{\partial}{\partial y}\right]$$

shows that the function $u(x,\ y)$ must satisfy a special differential equation of the third order. The family $u=c$ is consequently of special character. We may distinguish three classes of families of constant curvature: (1) flat families, which admit a parameter such that $K=0$; (2) spherical families, non-isothermal families which admit a *real* parameter such that $K=1$; (3) pseudo-spherical families, non-isothermal families which admit a *real* parameter such that $K=-1$. A family of constant curvature belongs to one and only one of these three classes. This fact is a consequence of (3.1) and the fact that a family $u=c$ for which $\Delta\equiv f(u)$ is isothermal.[3] The determination of all families of constant curvature is given in the following theorems.

THEOREM 8. *The orthogonal trajectories of any one-parameter family*

[3] Eisenhart, op. cit., p. 96.

of straight lines is a flat family. Conversely, every flat family can be reduced conformally to the orthogonal trajectories of some one-parameter family of straight lines.

THEOREM 9. *The orthogonal trajectories of any non-isothermal one-parameter family of circles of the form* $(x-c_1)^2+(y-c_2)^2=1+c_1^2+c_2^2$ *is a spherical family. Conversely, every spherical family can be reduced conformally to the orthogonal trajectories of some one-parameter family of such circles.*

THEOREM 10. *The orthogonal trajectories of any non-isothermal one-parameter family of circles of the form* $x^2+(y-c_1)^2=c_2^2$ *is pseudo-spherical. Conversely, every pseudo-spherical family can be reduced conformally to the orthogonal trajectories of some one-parameter family of such circles.*

The proofs of these theorems are quite similar. Consider, for example, Theorem 9. Let Σ denote the unit sphere $\xi^2+\eta^2+\zeta^2=1$. If Σ be referred to its minimal[4] lines, $\alpha=c$, $\beta=d$, then any real one-parameter family of its geodesics is given by $C\alpha\beta+(A-iB)\alpha+(A+iB)\beta-C=0$, where A, B, C are real functions of a real variable t. The equations $X+iY=\alpha$, $X-iY=\beta$ define a real conformal correspondence between Σ and the (X, Y)-plane, the family of geodesics of Σ corresponding to $C(X^2+Y^2)+2AX+2BY=C$. In general $C\neq 0$, so that the images of the geodesics of Σ are the circles

$$\left(X+\frac{A}{C}\right)^2+\left(Y+\frac{B}{C}\right)^2=1+\left(\frac{A}{C}\right)^2+\left(\frac{B}{C}\right)^2. \tag{3.2}$$

From Theorem 7 it follows that the parameter of the orthogonal trajectories of the circles (3.2) can be so chosen that the associated surface for these trajectories is isometric with Σ. This proves the first part of Theorem 9. Suppose, conversely, that we are given a spherical family $u=c$, referred to a parameter such that $K=1$, then Σ is the associated surface. By Theorem 7 the orthogonal trajectories of $u=c$ are conformal images of a family of geodesics of Σ. As before we may map Σ conformally on the (X, Y)-plane so that its geodesics go over into the circles (3.2). This mapping clearly induces a conformal correspondence between the (x, y)-plane and the (X, Y)-plane so that the orthogonal trajectories of $u=c$ go over into the circles (3.2). This completes the proof. The proofs of the two remaining theorems are entirely analogous.

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[4] Eisenhart, op. cit., p. 109.