ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

454. R. H. Bruck and T. L. Wade: *The number of independent components of the tensor* ${}_{[\alpha]}T_{i_1 \cdots i_p}$.

Let $T_{i_1 \cdots i_p} = \sum_\alpha {}_{[\alpha]}T_{i_1 \cdots i_p}$ represent the decomposition of an arbitrary tensor $T_{i_1 \cdots i_p}$ into tensors of various symmetry types, a type corresponding to each partition $[\alpha]$ of the indices $i_1 \cdots i_p$ (H. Weyl, *The Classical Groups*, pp. 96–136; T. L. Wade *Tensor algebra and Young's symmetry operators*, American Journal of Mathematics, vol. 63 (1941) pp. 645–657). J. A. Schouten (*Der Ricci-Kalkül*, chap. 7, §7) considers the problem of finding the number of independent scalar components of c_α of the tensor ${}_{[\alpha]}T_{i_1 \cdots i_p}$ and obtains expressions for c_α in terms of the dimension n of the coordinate system in the cases $p=2, 3, 4$, but the difficulties of his method become great for $p \geqq 5$. In this note it is shown that c_α is equal to r_α, the rank, recently defined by the authors, of the immanent tensor ${}_{[\alpha]}I^{i_1 \cdots i_p}_{j_1 \cdots j_p}$ (abstract 47-5-185). The general formulas for the r_α's in terms of n may be obtained directly from the character table of the symmetric group on p letters whenever this is available, as for $p \leqq 13$. Tables of $c_\alpha = r_\alpha$ are given for $p \leqq 6$. (Received October 1, 1941.)

455. A. D. Campbell: *On the application of an algebra of sets to group theory.*

In this paper is assumed the existence of a set S of abstract entities (including 0 and 1) whose subsets are used as coordinates for groups, subgroups, elements of groups, sets of elements, and neighborhoods (in the study of topological groups). The relation $\alpha \subset \beta$ means that the element (or set or group or neighborhood) α is contained in the set (or group or neighborhood) β. By (α, β) is meant the join of α and β, by $\alpha \cdot \beta$ the meet of α and β. The unit element is called 1, the null-set (or null-group) is called 0. By $\alpha\beta$ is meant the set consisting of all products of elements of α by elements of β (in this order). Obvious meanings are given to $\alpha - \beta$ and α^{-1}. It is agreed that $[\alpha_1, \alpha_2, \cdots, \alpha_n]$ shall mean the group generated by $\alpha_1, \alpha_2, \cdots, \alpha_n$ and that $0\alpha = \alpha 0 = 0$, $1\alpha = \alpha 1 = \alpha$, $\alpha \cdot 0 = 0$, $\alpha - 0 = \alpha$. It is to be noted that $(\alpha, \beta)\gamma = (\alpha\gamma, \beta\gamma)$ and $(\alpha - \beta)\gamma = \alpha\gamma - \beta\gamma$ and $(\alpha \cdot \beta) \cdot \gamma = (\alpha\gamma) \cdot (\beta\gamma)$; also that for a group $\alpha\alpha = \alpha$, $\alpha\alpha^{-1} = \alpha$, $\alpha^{-1} = \alpha$. By this algebra old theorems are readily proved and new ones are derived. Thus it follows that if $\alpha = (\beta, \gamma)$ is a finite group of order a with β as a subgroup of order b and with $\beta \cdot \gamma = 0$, then the relation $\beta\alpha = (\beta\beta, \beta\gamma) = (\beta, \beta\gamma)$ shows clearly (since $\beta \cdot (\beta\gamma) = 0$) the well known result that $a = b + bc = b(1+c)$. (Received September 15, 1941.)

456. Tomlinson Fort: *Generalizations of the Bernoulli polynomials and numbers and corresponding summation formulas.*

The operators d/dx and $\sum$ can be used to define the Bernoulli polynomials. These are generalized to linear operators by means of which very general classes of polynomials are defined, the coefficients of which serve to generalize the Bernoulli numbers. The polynomials determined are very general and include as special cases not only the classical Bernoulli polynomials, but the Bernoulli polynomials of the second kind, the Bernoulli and Euler polynomials of higher order as defined by Nörlund (*Differenzenrechnung*, p. 119) and the Bernoulli numbers as generalized by Vandiver (Proceedings of the National Academy of Sciences, vol. 23, p. 555). Attention is called to several new and interesting classes of polynomials and corresponding numbers. The polynomials of the paper are made the starting point for summation formulas analogous to Taylor's formula and to the Euler-Maclaurin formula of classical mathematics. A form for the remainder is obtained analogous to the Lagrange form for the remainder in Taylor's formula. (Received September 18, 1941.)

457. D. H. Lehmer: *Properties of the coefficients of the modular invariant $J(\tau)$.*

The properties of the integer coefficients in the Fourier series development of Klein's absolute elliptic modular invariant $j(\tau)=12^3J(\tau)=e^{-2\pi i\tau}+744+196884e^{2\pi i\tau}+\cdots=\sum_{n=-1}^{\infty}c(n)\exp(2\pi in\tau)$ are of two sorts: (a) congruence properties with respect to small moduli and (b) multiplicative properties. The congruence properties are similar to those of Ramanujan's numerical function $\tau(n)$ with which the c's are intimately related. For example, $(k+1)c(k)\equiv 0 \pmod{24}$. Multiplicative properties are discovered which enable one to express $c(kn)$ simply in terms of previous c's for $n<k+1$. For example, $c(10)=c(6)+c(1)c(4)+c(2)c(3)$. Similar properties hold for the coefficients of any positive integral power of $j(\tau)$. (Received September 30, 1941.)

458. Howard Levi: *A characterization of polynomial rings by means of order relations.*

In this paper rings are considered for whose elements certain order relations and algorithms have been assumed. The assumptions made were borrowed from polynomial rings, where they appear as fairly obvious facts. It is shown that these facts characterize polynomial rings; that any ring for which they are postulated must be a polynomial ring. In view of the fact that polynomial rings with different types of coefficient domains differ significantly, two lists of assumptions are presented. The first leads to a ring of polynomials whose cofficients constitute a domain of integrity which contains a unit element and in which the Hilbert basis theorem holds. The coefficient domain derived with the second list is a field. These two lists of assumptions differ only slightly, and minor modifications of them could be made which would lead to more general types of coefficients. These cases are not discussed, in the belief that a polynomial whose cofficients are of a very general type has few claims to attention which it does not share with any element of a general ring. (Received September 25, 1941.)

459. R. J. Levit: *Fields in terms of a single operation.*

Though it is customary to consider fields as based on two independent operations, they may equally well be regarded as systems of single composition. This was first shown by N. Wiener when he exhibited a set of seven postulates for fields in terms of a single operation. In the present paper are presented two other definitions of fields each in terms of a single operation and requiring five and six postulates respectively. The former takes as primitive an operation expressible as $a(1-b)$, which has the advantage of being class-closing. The latter definition uses Wiener's operation interpreted as $1-a/b$, which, though not class-closing, makes it possible to dispense with all existence postulates outside of the closure conditions. Detailed proofs are supplied of the necessity, sufficiency, consistency, and independence of both sets. A number of other single operations that can be used to define fields are also discussed. It is shown that no field-defining operation (or set of operations) is uniquely characterized by the postulates it satisfies. Each such operation (or set) is susceptible of a variety of interpretations, and all such interpretations are found. (Received October 1, 1941.)

460. Ernst Snapper: *Co-maximal linear sets and products of linear sets.*

The linear subsets of an n-dimensional vector space V_n satisfy the Noether decomposition theory (this Bulletin, abstract 47-3-124). The following three definitions are introduced: The product of two linear sets L_1 and L_2 is $(\mathfrak{E}_1L_2+\mathfrak{E}_2L_1)$, where $\mathfrak{E}_i$ is the essential ideal of L_i. Two linear sets are co-maximal if their sum is the whole vector space. Two linear sets are strongly co-maximal if the sum of their essential ideals is the whole scalar domain. Strongly co-maximal linear sets are co-maximal. The definitions become the familiar definitions of co-maximal ideals and products of ideals if $n=1$. The following theorems are proved: The intersection of a finite number of linear sets, strongly co-maximal in pairs, equals their product. Every linear set is the irredundant intersection of a finite number of linear sets, strongly co-maximal in pairs; the intersection components themselves are not irredundant intersections of strongly co-maximal pairs and are unique. The results are used to study the decomposition of the restclass group V_n/L into a direct sum of subgroups. (Received September 8, 1941.)

461. L. R. Wilcox: *Extensions of semi-modular lattices.* II.

This paper generalizes a theorem of the author concerning complemented semi-modular lattices L (abstract 47-5-208) to the case where L is of dimension greater than or equal to 4 (that is, there exists in L a chain $a_1>a_2> \cdots >a_n$, where $n\geqq 6$), and L possesses no two parallel hyperplanes. (Received August 7, 1941.)

462. R. E. O'Connor: *A theorem of Diophantine approximation and an application to the values of a linear form.*

Assume θ given, $\eta\geqq 1$, x, y rational integers and all other numbers real. Let the proposition P_1 assert that $c=c(\theta)$ exists such that, for every $t>1$ and every α, the non-homogeneous system, $0<x<ct^\eta$, $|x\theta-y-\alpha|<t^{-1}$, has a solution (x, y). Let P_2 assert that a positive γ exists such that, for every $t>1$, the homogeneous system, $0<x<\gamma t^{1/\eta}$, $|x\theta-y|<t^{-1}$, is insoluble. Referring to the standard classification of irrationals (see Koksma, *Diophantische Approximationen*, p. 27), it is easily shown that *P_2 is equivalent to the proposition that θ is irrational and of type lower than $I\eta\,\infty$*. In the present paper it is established that *P_1 and P_2 are equivalent propositions*, a theorem proved in 1924 by A. Khintchine for the case $\eta=1$. Now let ω, ω' be positive numbers

with ratio θ and let $\mu_0=0$, μ_1, $\mu_2, \cdots$ be the sequence S_μ of values, in order of increasing magnitude, taken by the form $x\omega+y\omega'$ as x, y range through the non-negative integers. For every positive μ, define $\Delta(\mu)=\max_{\mu_{i+1}>\mu}(\mu_{i+1}-\mu_i)$. Employing the above theorems it is proved that $\Delta(\mu)=O(\mu^{-1/\eta})$ $(\mu\to\infty)$, *if and only if* θ *is irrational and of type lower than* $I\eta\infty$. Further information concerning the sequence S_μ is also given and extensions of the last theorem to cases with θ an irrational of higher type. (Received August 21, 1941.)

463. R. E. O'Connor: *Representation of integers by power-products of two integers.*

With a, b integers, $1<a<b$, let $A(n)$ be the number of representations of the positive integer n in the form $n=\sum c_{ij}a^ib^j$, the coefficients being non-negative integers less than a. Then $A(n)$ *is equal to* $P(n)=P(b, n)$, the number of partitions of n into non-negative powers of b. By a modification of a method of Hardy and Ramanujan (Proceedings of the London Mathematical Society, (2), vol. 17 (1918), pp. 75–115) it is shown that $\log P(n)\sim(\log n)^2/2\log b$, *as* $n\to\infty$. The same result is established by showing, with $f(x)=1+\sum_{n=1}^{\infty}P(n)x^n(|x|<1)$, that $\log f(x)\sim(\log^2(1-x))/2\log b$ $(x\to 1-0)$ and by using this asymptotic value of $\log f(x)$ in the following theorem of Tauberian type. *Assume*: (1) $\phi(t)=a_0+a_1e^{-t}+a_2e^{-2t}+\cdots$, $a_i\geqq 0$ $(i=0, 1, 2, \cdots)$, *converges for* $t>0$; (2) $\log\phi(t)=(1+o(1))K(\log(t^{-1}))^\alpha$ $(t\to 0)$, $K>0$, $\alpha>0$; (3) $A_n=\sum_{i=0}^{n}a_i$ $(n=0, 1, 2, \cdots)$. Then $\log A_n=(1+o(1))K(\log n)^\alpha$ $(n\to\infty)$. (Received August 21, 1941.)

464. R. E. O'Connor: *Representation of integers by power-products of two real numbers.*

Let α, β *be real numbers greater than unity with* $\theta=\log\alpha/\log\beta$ *irrational.* Define $M(n)$ as the greatest integer such that every integer M', $0<M'\leqq M$, has a representation of the form $M'=[\alpha^{r_1}\beta^{s_1}]+[\alpha^{r_2}\beta^{s_2}]+\cdots+[\alpha^{r_\nu}\beta^{s_\nu}]$ where the r_i, s_i are non-negative integers and $\nu\leqq n$. Very simple considerations show that *to any* $\epsilon>0$ *there corresponds an* $n_0=n_0(\epsilon, \alpha, \beta)$ *such that* $\log M(n)<(2+\epsilon)n\log n$, *for* $n>n_0$. Lower bounds for $M(n)$ are obtained in two cases as follows. *Assume* θ *is of type lower than* $I\eta\infty$; *then for every positive* k *there is an* $n_0=n_0(k, \alpha, \beta)$ *such that* $\log M(n)>\eta^{-1}n\log n+kn$ $(n>n_0)$. Or, *assume* α, β *algebraic; then for every positive* ϵ *there is an* $n_0=n_0(\epsilon, \alpha, \beta)$ *such that* $\log M(n)>n(\log n)^{1/3-\epsilon}$ $(n>n_0)$. In establishing the first of these lower bounds use is made of the theorems of abstract 47-11-462; for the second is required a theorem of Gelfond (Bulletin de l'Académie des Sciences de l'URSS, Série Mathématique, 1939, pp. 509–518). (Received August 21, 1941.)

465. Gordon Pall: *An elementary alternative to Dirichlet's theorem.*

It is shown how the application of Dirichlet's theorem on primes in an arithmetical progression can be replaced in many problems (especially concerning quadratic forms) by the application of Gauss's theorem on the existence of binary quadratic forms with assigned values for their generic invariants. For example, Meyer's theorem on indefinite quinary forms representing zero and the theorem giving conditions for quaternary zero forms can be proved by entirely elementary methods. (Received August 7, 1941.)

466. H. S. Vandiver: *An arithmetical theory of Bernoulli numbers.*

In this paper a method is described which enables one to find many new types of arithmetical relations concerning the Bernoulli and allied numbers. It depends mainly on the following idea: Let a and b be rational, with $a \equiv b \pmod{p}$, where p is any prime integer. If a and b are independent of p it then follows, since there is an infinity of primes, that $a = b$. This is applied in connection with wha is perhaps the simplest formula in which a single Bernoulli number appears: $S_n^{(} \equiv pb_n \pmod{p^2}$, where $S_n(p) = 1^n + 2^n + \cdots + (p-1)^n$. The paper starts with simple identities involving $(x^p - 1)/(x-1)$, obtains congruences modulo p or p^2 by differentiation and integration, and then makes substitutions for the variables which give congruences involving the Bernoulli numbers. Generalizations of the Staudt-Clausen theorem as well as analogues of the same are obtained. Proofs are given of nearly all the results which were stated without proof in two previous papers by the writer (Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 555–559; vol. 25 (1939), pp. 197–201). This article will appear in the Transactions of this Society. (Received August 12, 1941.)

Analysis

467. Lipman Bers: *A convergence theorem for analytic functions of two variables.*

Let M be a domain of the four-dimensional z_1, z_2-space $(z_k = x_k + iy_k, k = 1, 2)$ bounded by the two analytic hypersurfaces $E[z_1 \epsilon B(z_2) + c(z_2),\ z_2 = e^{i\theta},\ 0 \leqq \theta < 2\pi]$ and $E[z_1 \epsilon c(z_2),\ |z_2| < 1]$ where $B(z_2)$ is a star domain bounded by the curve $c(z_2) = E(z_1 = h(z_2, \lambda),\ 0 \leqq \lambda < 2\pi]$ and $h(z_2, \lambda)$ is an analytic function of z_2, $|z_2| \leqq 1$, for every fixed value of λ, and is subject to certain additional conditions. Consider a sequence of analytic functions $\{f_n(z_1, z_2)\}$ defined in M and satisfying the condition $\int_0^{2\pi}\int_0^{2\pi} |f[r_1 h(r_2 e^{i\theta}, \lambda), r_2 e^{i\theta}]|^p d\theta d\lambda < K$, $0 < r_k < 1$, $k = 1, 2$, $n = 1, 2, \cdots$, where p is a fixed number greater than 1. It is known that $f_n(z_1, z_2)$ possesses finite sectorial limits $(F_n(\theta, \lambda)$ almost everywhere on the intersection $F = E[z_1 = h(e^{i\theta}, \lambda),\ z_2 = e^{i\theta},\ 0 \leqq \theta < 2\pi,\ 0 \leqq \lambda \leqq 2\pi]$ of the two boundary hypersurfaces of M (see Bergman and Marcinkiewicz, Fundamenta Mathematicae, vol. 33 (1939), pp. 75–94, and a paper by the author to appear in the American Journal of Mathematics). If the sequence $\{F_n(\theta, \lambda)\}$ converges in a set of positive two-dimensional measure, the sequence $\{f_n(z_1, z_2)\}$ converges uniformly in every closed subdomain of M. This theorem can be generalized, in a modified form, for more general types of domains. (Received September 27, 1941.)

468. A. B. Brown: *Independent parameters for sets of functions.*

The results previously announced for the case of one function of n variables and m parameters (under the title *On the number of independent parameters*, abstract 46-11-485) are now extended to the case of a set of r functions of n variables and m parameters. The methods and results are similar to those for the case of one function. An additional theorem is given, on invariance of the number of parameters in a "complete" set under certain transformations of parameters. Only one paper is offered for publication, covering the more general case announced here, but under the original title. (Received September 11, 1941.)

469. E. R. Lorch: *The theory of analytic functions in normed abelian vector rings.* Preliminary report.

A complex vector space $\mathfrak{R}$ for which the commutative multiplication of elements is defined is here called a normed abelian vector ring if the norm satisfies