OSCULATING QUADRICS OF RULED SURFACES IN RECIPROCAL RECTILINEAR CONGRUENCES

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1. **Introduction.** Let x be a general point of an analytic non-ruled surface S referred to its asymptotic net in ordinary projective space. By a line l_1 at the point x we mean any line through the point x and not lying in the tangent plane of the surface at the point x. Dually, a line l_2 is any line in the tangent plane of the surface at the point x but not passing through the point x. The lines l_1, l_2 are called reciprocal lines if they are reciprocal polar lines with respect to the quadric of Lie at the point x. In this case, when the point x varies over the surface S, the lines l_1, l_2 generate two rectilinear congruences Γ_1, Γ_2 which are said to be reciprocal with respect to the surface. If, however, the point x moves along the u-curve, the locus of the line l_1 is a ruled surface $R_1^{(u)}$ of the congruence Γ_1. The osculating quadric along a generator l_1 of the ruled surface $R_1^{(u)}$ is the limit of the quadric determined by the line l_1 through the point x and the lines l_1 through two neighboring points P_1, P_2 on the u-curve as each of these points independently approaches the point x along the u-curve. The quadric thus defined will be denoted by $Q_1^{(u)}$. A second quadric $Q_1^{(v)}$ is determined by three consecutive lines l_1 at points of the v-curve through the point x. Moreover, there are two quadrics, denoted by $Q_2^{(u)}$ and $Q_2^{(v)}$, which are associated with two ruled surfaces of the reciprocal congruence Γ_2 and which can be defined similarly. This note will study the projective differential geometry of the quadrics thus defined.

2. **Analytic basis.** Let the surface S under consideration be an analytic non-ruled surface whose parametric vector equation, referred to asymptotic parameters u, v, is

$$x = x(u, v). \tag{1}$$

The four coordinates x of a variable point x on the surface satisfy two partial differential equations which can be reduced, by a suitably chosen transformation of proportionality factor, to Fubini's canonical form

$$x_{uu} = px + \theta_u x_u + \beta x_v, \qquad x_{vv} = qx + \gamma x_u + \theta_v x_v, \qquad \theta = \log \beta\gamma, \tag{2}$$

in which the subscripts indicate partial differentiation. The coefficients of these equations are functions of u, v and satisfy three integrability conditions which need not be written here.

Two lines l_1, l_2 are reciprocal lines if the line l_1 joins the point x and the point y defined[1] by

$$(3) \qquad y = -ax_u - bx_v + x_{uv}$$

and the line l_2 joins the points ρ, σ defined by placing

$$(4) \qquad \rho = x_u - bx, \qquad \sigma = x_v - ax,$$

where a, b are functions of u, v. As u, v vary, the lines l_1, l_2 generate two rectilinear congruences Γ_1, Γ_2 which are reciprocal with respect to the surface.

The curves corresponding to the developables of the congruence Γ_1 are called the Γ_1-curves of the congruence, and those corresponding to the developables of the congruence Γ_2 the Γ_2-curves of the congruence. The differential equation of the Γ_1-curves is

$$(5) \qquad (F - 2a\beta + \beta\psi)du^2 - (b_v - a_u)dudv - (G - 2b\gamma + \gamma\phi)dv^2 = 0,$$

where F, G are defined by the formulas

$$F = p - b_u + b\theta_u - b^2 + a\beta, \qquad G = q - a_v + a\theta_v - a^2 + b\gamma,$$

and ϕ, ψ by

$$\phi = (\log \beta\gamma^2)_u, \qquad \psi = (\log \beta^2\gamma)_v.$$

If k_1, k_2 are the roots of the equation

$$(6) \qquad k^2 + (A + B)k + AB - (F - 2a\beta + \beta\psi)(G - 2b\gamma + \gamma\phi) = 0,$$

where the functions A, B are defined by

$$A = -a_u - ab + \beta\gamma + \theta_{uv}, \qquad B = -b_v - ab + \beta\gamma + \theta_{uv},$$

the corresponding points

$$y + k_i x, \qquad i = 1, 2,$$

are the focal points of the line l_1. Furthermore, the differential equation of the Γ_2-curves is

$$(7) \qquad Fdu^2 - (b_v - a_u)dudv - Gdv^2 = 0.$$

If τ_1, τ_2 are the roots of the equation

$$(8) \qquad F + (b_v - a_u)\tau - G\tau^2 = 0,$$

the corresponding points

$$\rho + \tau_i\sigma, \qquad i = 1, 2,$$

[1] In this section we employ the notation used by E. P. Lane in Chapter III of his book *Projective Differential Geometry of Curves and Surfaces*, Chicago, 1932.

are the focal points of the line l_2. It will be assumed that the coefficients of du^2 and dv^2 in equations (5), (7) are all nonzero. In this case the Γ_1-curves and the Γ_2-curves of two reciprocal congruences form conjugate nets if, and only if, $b_v - a_u = 0$.

3. **Osculating quadrics of ruled surfaces of the congruence Γ_1.** Any point z, except the point y, on the line l_1 at the point x is given by the equation

$$z = x + \omega y, \qquad \omega \text{ scalar.} \tag{9}$$

As the point x varies along the u-curve, the line l_1 generates a ruled surface $R_1^{(u)}$. Equation (9) is the parametric vector equation of this ruled surface, u, ω being the independent parameters. The asymptotic curves on $R_1^{(u)}$ consist of the lines l_1 and the integral curves of the differential equation

$$L_1 du + 2M_1 d\omega = 0, \tag{10}$$

where L_1, M_1 are determinants of the fourth order defined by

$$L_1 = (z_{uu}, z, z_u, z_\omega), \qquad M_1 = (z_{u\omega}, z, z_u, z_\omega).$$

Differentiating equation (9) and using equations (2), (3) to calculate the values of L_1, M_1, we find that equation (10) can be written in the form

$$\frac{d\omega}{du} = -\frac{\beta + C\omega + D\omega^2}{2(F - 2a\beta + \beta\psi)}, \tag{11}$$

where we have placed

$$\begin{aligned} C &= F_u - 2(a\beta)_u + (\beta\psi)_u + 2\beta A, \\ D &= \beta A^2 - a(F - 2a\beta + \beta\psi)^2 + A(F - 2a\beta + \beta\psi)_u \\ &\quad - (F - 2a\beta + \beta\psi)[p_v + \beta q - ap + A(b - \theta_u) + A_u]. \end{aligned}$$

Any point X, except the point z, on the tangent at the point z of the curved asymptotic on $R_1^{(u)}$ is defined by placing

$$X = \lambda z + dz/du, \qquad \lambda \text{ scalar.} \tag{12}$$

If we use the tetrahedron x, ρ, σ, y as a local tetrahedron of reference with a unit point chosen so that a point

$$x_1 x + x_2 \rho + x_3 \sigma + x_4 y$$

has local coordinates proportional to $x_1, \cdots, x_4$, we find that the local coordinates of the point X are given by

$$(13)\qquad \begin{aligned} x_1 &= b + \lambda + [a(F - 2a\beta + \beta\psi) + bA + p_v + \beta q - ap]\omega, \\ x_2 &= 1 + A\omega, \\ x_3 &= (F - 2a\beta + \beta\psi)\omega, \\ x_4 &= \omega\lambda - (b - \theta_u)\omega - \frac{\beta + C\omega + D\omega^2}{2(F - 2a\beta + \beta\psi)} \cdot \end{aligned}$$

Homogeneous elimination of ω, λ from these equations gives *the algebraic equation of the quadric $Q_1^{(u)}$, referred to the tetrahedron x, ρ, σ, y,* namely

$$(14)\qquad \begin{aligned} &\beta(F - 2a\beta + \beta\psi)x_2^2 + Hx_3^2 + 2(F - 2a\beta + \beta\psi)^2 x_2x_4 \\ &\quad - 2A(F - 2a\beta + \beta\psi)x_3x_4 \\ &\quad - 2(F - 2a\beta + \beta\psi)x_1x_3 + 2Px_2x_3 = 0, \end{aligned}$$

where the coefficients H, P are defined by

$$(15)\qquad \begin{aligned} H &= a(F - 2a\beta + \beta\psi) - 3A(b - \theta_u) - A_u + p_v + \beta q - ap, \\ P &= (2b - \theta_u)(F - 2a\beta + \beta\psi) + \tfrac{1}{2}(F - 2a\beta + \beta\psi)_u. \end{aligned}$$

The equation of the quadric $Q_1^{(v)}$ can be written by interchanging u and v and making the appropriate symmetrical interchanges of the other symbols. The result is

$$(16)\qquad \begin{aligned} &Kx_2^2 + \gamma(G - 2b\gamma + \gamma\phi)x_3^2 + 2(G - 2b\gamma + \gamma\phi)^2 x_3x_4 \\ &\quad - 2B(G - 2b\gamma + \gamma\phi)x_2x_4 \\ &\quad - 2(G - 2b\gamma + \gamma\phi)x_1x_2 + 2Qx_2x_3 = 0, \end{aligned}$$

where the coefficients K, Q are given by

$$(17)\qquad \begin{aligned} K &= b(G - 2b\gamma + \gamma\phi) - 3B(a - \theta_v) - B_v + q_v + \gamma p - bq, \\ Q &= (2a - \theta_v)(G - 2b\gamma + \gamma\phi) + \tfrac{1}{2}(G - 2b\gamma + \gamma\phi)_v. \end{aligned}$$

Some properties of the quadrics $Q_1^{(u)}$, $Q_1^{(v)}$ will now be deduced. In the first place, the tangent plane, $x_4 = 0$, intersects each of the quadrics in a conic. The conic of intersection of the tangent plane and the quadric $Q_1^{(u)}$ touches the u-tangent at the point x and intersects the v-tangent in the point whose local coordinates are

$$(18)\qquad (\tfrac{1}{2}H,\ 0,\ F - 2a\beta + \beta\psi,\ 0).$$

Similarly, the quadric $Q_1^{(v)}$ is intersected by the tangent plane in a conic which is tangent to the v-tangent at the point x and intersects the u-tangent in the point

(19) $$(\tfrac{1}{2}K, G - 2b\gamma + \gamma\phi, 0, 0).$$

The face $x_3=0$ of the tetrahedron of reference intersects the quadric $Q_1^{(u)}$ in the line l_1 and in the line whose equations are

(20) $$\beta x_2 + 2(F - 2a\beta + \beta\psi)x_4 = 0, \qquad x_3 = 0.$$

Moreover, the face $x_2=0$ cuts the quadric $Q_1^{(u)}$ in the line l_1 and in the line which joins the point (18) to the point on the line l_1 with local coordinates

(21) $$(-A, 0, 0, 1).$$

Similarly, the face $x_2=0$ cuts the quadric $Q_1^{(v)}$ in the line l_1 and in the line

(22) $$\gamma x_3 + 2(G - 2b\gamma + \gamma\phi)x_4 = 0, \qquad x_2 = 0.$$

The face $x_3=0$ cuts the quadric $Q_1^{(v)}$ in the line l_1 and in the line which passes through the point (19) and meets the line l_1 in the point

(23) $$(-B, 0, 0, 1).$$

The points (21), (23) are found to coincide if, and only if, $a_u=b_v$. Thus we reach the following conclusion:

The Γ_1-curves and the Γ_2-curves of two reciprocal rectilinear congruences form conjugate nets on the surface if, and only if, the points (21), (23) *coincide.*

It is well known that two nonsingular quadric surfaces having one, and only one, generator in common intersect elsewhere in a twisted cubic. Elimination of x_1 between equations (14), (16) gives the cubic cone projecting the curve of intersection of the two quadrics from the point x. This cone has the line l_1 for a double line, the equations of the nodal tangent planes along the line l_1 being given by

(24) $$(F - 2a\beta + \beta\psi)x_2^2 - (b_v - a_u)x_2x_3 - (G - 2b\gamma + \gamma\phi)x_3^2 = 0.$$

A glance at equation (5) suffices to substantiate the following statement:

The nodal tangent planes along the double line l_1 of the cone projecting the curve of intersection of the quadrics $Q_1^{(u)}$, $Q_1^{(v)}$ from the point x are the planes which intersect the tangent plane of the surface at the point x in the tangents of the Γ_1-curves.

Eliminating x_2 from equations (14), (16), we obtain the equation of the cone which projects the curve of intersection of the quadrics

$Q_1^{(u)}$, $Q_1^{(v)}$ from the vertex ρ of the tetrahedron of reference. This projecting cone is found to be a composite quartic cone, one component being the face $x_3=0$ of the tetrahedron of reference. The other component is a cubic cone which is intersected by the face $x_2=0$ in a plane cubic curve. Placing $x_3=0$ in the equation of this curve, we find the intersections of the curve with the line l_1. It is now easy to verify the conclusion:

The quadrics $Q_1^{(u)}$, $Q_1^{(v)}$ intersect in the line l_1 and in a twisted cubic which crosses the line l_1 in its two focal points.

4. **Osculating quadrics of ruled surfaces of the congruence Γ_2.** The equations of the quadrics $Q_2^{(u)}$, $Q_2^{(v)}$ can be found without difficulty by applying the method of the preceding section. The details of the calculation need not be reproduced here, but the required *equation of the quadric $Q_2^{(u)}$, referred to the tetrahedron x, ρ, σ, y,* is found to be

$$\beta x_1^2 - 2Fx_1x_3 + 2F^2x_2x_4 + 2(ab - a_u)Fx_3x_4 + 2Sx_1x_4 + Lx_4^2 = 0, \tag{25}$$

where the functions S, L are defined by

$$\begin{aligned} S &= \tfrac{1}{2}F_u - F\theta_u + 2bF - \beta(ab - a_u), \\ L &= \beta FG + FF_v - F(\theta_u - 2b)(b_v - a_u) + F(b_v - a_u)_u \\ &\quad - 2S(ab - a_u) - \beta(ab - a_u)^2. \end{aligned} \tag{26}$$

The equation of the quadric $Q_2^{(v)}$ is

$$\gamma x_1^2 - 2Gx_1x_2 + 2G^2x_3x_4 + 2(ab - b_v)Gx_2x_4 + 2Tx_1x_4 + Mx_4^2 = 0 \tag{27}$$

where

$$\begin{aligned} T &= \tfrac{1}{2}G_v - G\theta_v + 2aG - \gamma(ab - b_v), \\ M &= \gamma FG + GG_u - G(\theta_v - 2a)(a_u - b_v) + G(a_u - b_v)_v \\ &\quad - 2T(ab - b_v) - \gamma(ab - b_v)^2. \end{aligned} \tag{28}$$

The quadric $Q_2^{(u)}$ is intersected by the tangent plane in the line l_2 and also in the line

$$\beta x_1 - 2Fx_3 = 0, \qquad x_4 = 0. \tag{29}$$

The face $x_3=0$ cuts the quadric $Q_2^{(u)}$ in a conic which is tangent to the u-tangent at the point ρ and which intersects the edge $x_1=x_3=0$ in the point with local coordinates

$$(0,\ L,\ 0,\ -2F^2). \tag{30}$$

The face $x_1=0$ cuts the quadric $Q_2^{(u)}$ in the line l_2 and in the line which joins the point

$$(31) \qquad (0,\ a_u - ab,\ F,\ 0)$$

on the line l_2 to the point (30).

Similarly, the tangent plane intersects the quadric $Q_2^{(v)}$ in the line l_2 and in the line

$$(32) \qquad \gamma x_1 - 2Gx_2 = 0, \qquad x_4 = 0.$$

The plane $x_2=0$ cuts this quadric in a conic which is tangent to the v-tangent at the point σ and which intersects the edge $x_1=x_2=0$ in the point

$$(33) \qquad (0,\ 0,\ M,\ -2G^2).$$

The face $x_1=0$ intersects the quadric $Q_2^{(v)}$ in the line l_2 and in the line which joins the point

$$(34) \qquad (0,\ G,\ b_v - ab,\ 0)$$

on the line l_2 to the point (33). The following conclusion is immediate.

If the points (31), (34) *coincide respectively with the points* σ, ρ, *the* Γ_1-*curves and the* Γ_2-*curves form conjugate nets.*

Elimination of x_2 from equations (25), (27) yields the equation of the cubic cone projecting from the point ρ the curve of intersection of the quadrics $Q_2^{(u)}$, $Q_2^{(v)}$. The line l_2 is a double line of this cone, the nodal tangent planes along the line l_2 being given by

$$(35) \qquad x_1^2 - (b_v - a_u)x_1x_4 - [FG + (ab - b_v)(ab - a_u)]x_4^2 = 0.$$

It is now easy to verify the conclusion:

The two nodal tangent planes along the double line l_2 *of the cone projecting the curve of intersection of the quadrics* $Q_2^{(u)}$, $Q_2^{(v)}$ *from the point* ρ *intersect the line* l_1 *in two points which separate the points* x, y *harmonically if, and only if, the* Γ_1-*curves and the* Γ_2-*curves form conjugate nets.*

Finally, simple calculations suffice to demonstrate the following theorem:

The quadrics $Q_2^{(u)}$, $Q_2^{(v)}$ *intersect in the line* l_2 *and in a twisted cubic which cuts the line* l_2 *in its two focal points.*

5. **A special case.** The theory of the preceding sections will now be specialized by considering a particular covariant pair of reciprocal

lines associated with the point x of the surface. It is known that the line l_1 is the projective normal and the line l_2 is the reciprocal projective normal in case $a=b=0$ in equations (3), (4). Placing $a=b=0$ in equations (14), (16), one easily shows that the equations of the two quadrics $Q_1^{(u)}$, $Q_1^{(v)}$, which we shall call *the projective normal quadrics*, are respectively

$$(36)\quad \begin{aligned} &\beta\pi x_2^2 + (p_v + \beta q + 3\theta_u k - k_u)x_3^2 + 2\pi^2 x_2x_4 + (\pi_u - 2\pi\theta_u)x_2x_3 \\ &\qquad - 2\pi k x_3x_4 - 2\pi x_1x_3 = 0, \\ &\gamma\chi x_3^2 + (q_u + \gamma p + 3\theta_v k - k_v)x_2^2 + 2\chi^2 x_3x_4 + (\chi_v - 2\chi\theta_v)x_2x_3 \\ &\qquad - 2\chi k x_2x_4 - 2\chi x_1x_2 = 0, \end{aligned}$$

where π, χ, k are defined by the formulas

$$\pi = p + \beta\psi, \qquad \chi = q + \gamma\phi, \qquad k = \beta\gamma + \theta_{uv}.$$

Moreover, by placing $a=b=0$ in equations (25), (27), we obtain *the two reciprocal projective normal quadrics* $Q_2^{(u)}$, $Q_2^{(v)}$, whose equations are respectively

$$(37)\quad \begin{aligned} &\beta x_1^2 + p(p_v + \beta q)x_4^2 + (p_u - 2p\theta_u)x_1x_4 - 2px_1x_3 + 2p^2 x_2x_4 = 0, \\ &\gamma x_1^2 + q(q_u + \gamma p)x_4^2 + (q_v - 2q\theta_v)x_1x_4 - 2qx_1x_2 + 2q^2 x_3x_4 = 0. \end{aligned}$$

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