REPRESENTATIONS OF BOOLEAN ALGEBRAS

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There are several proofs in the literature[1] of M. H. Stone's theorem on the representation of Boolean algebras by sets [**2**, **4**, **5**, **7**, **8**, **9**]. This note contains a simplified version of Stone's original proof, adapted to the following set, I–IV, of postulates for a Boolean algebra B in terms of the special element 0 and the undefined operations *product* ab and *negation* b'. It is assumed that 0 is in B, and that if a, b, and c are in B, then ab and b' are in B, and

I. $ab=ba$.

II. $a(bc)=(ab)c$.

III. $aa=a$.

IV. $ab=a$ if and only if $ab'=0$.

Replacing b by a in IV gives V: $aa'=0$. Since $a0=a(aa')=(aa)a'=aa'=0$, we have VI: $a0=0$.

DEFINITIONS. *A point is a set P of elements of B such that*

α. *The element* 0 *is not in P.*

β. *If a is in P and b is in P, then ab is in P.*

γ. *P is maximal with respect to properties α and β.*

The set R_a of all points P which contain a is defined to be the representative set corresponding to the element a.

LEMMA 1. *If ab is in P, then a is in P.*

PROOF. If a were not in P, then P would not be maximal, since a and all products pa, where p is in P, could then be added to P without disturbing α, since if $pa=0$, then $pab=0$.

LEMMA 2. *If a is not equal to* 0, *then a is in some point P.*

PROOF. All sets of elements of B which contain a and satisfy α and β form a system S partially ordered by set inclusion. Any linearly ordered subsystem L of S has an upper bound in S, namely the union of all members of L. Hence by Zorn's lemma [**10**, **11**], there exists in S at least one maximal element P.

THEOREM. *The correspondence between elements a of B and their representative sets R_a is an isomorphism; that is,* 1. $R_{ab}=R_a\cap R_b$; 2. $R_{a'}=C(R_a)$; 3. *if* $R_a=R_b$, *then* $a=b$.

[1] See also N. H. McCoy and D. Montgomery, Duke Mathematical Journal, vol. 3 (1937), pp. 455–459.

PROOF. 1. If P is in R_a and R_b, then it is in R_{ab} by β. Conversely, if P is in R_{ab}, it is in R_a and R_b by Lemma 1.

2. R_a and $R_{a'}$ are complementary, for if a point P is not in $R_{a'}$, there is an element b in P such that $a'b=0$, since otherwise a' and products $a'p$ could be added to P, and P would not be maximal. Hence by IV, $ab=b$; therefore ab is in P. Then by Lemma 1, a is in P, and P is in R_a. On the other hand, R_a and $R_{a'}$ have no common point P, since such a P would have to contain $aa'=0$.

3. If $a\neq b$, then either $ab'\neq 0$ or $a'b\neq 0$, since otherwise by IV $a=ab=b$. If $ab'\neq 0$, then by Lemma 2, ab' is in some point P. By Lemma 1, a is in P. But b is not in P, since $ab'b=0$. Likewise if $a'b\neq 0$, there is a point containing b but not a. Hence $R_a\neq R_b$.

COROLLARY. *The set* I–IV *is an adequate postulate system for Boolean algebras.*

For I–IV hold in any Boolean algebra, and any algebra in which they hold has been shown to be isomorphic to an algebra of sets, and hence to a Boolean algebra. This postulate system is comparable in simplicity with other well known sets [**1**, **3**, **6**].

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