$\sum_{i=1}^{n} a_i^2 + r\sum_{i=1}^{n} b_i^2$ (n any positive integer, $r=1, 2, 3$) are obtained. Other values of r can be treated by extending the function $G(x, t)$. This has been done for $r=4, 5, 7$, but the expansion for general r has not been obtained. The method also has been used to treat certain related forms. (Received July 29, 1941.)

361. R. D. James: *On the sieve method of Viggo Brun.*

A. A. Buchstab (Matematicheskii Sbornik, vol. 46 (1938), pp. 375–387, and Comptes Rendus de l'Académie des Sciences, URSS, vol. 29 (1940), pp. 544–548) has made improvements in the sieve method and so obtained more precise results. It is the purpose of this paper to point out that his method applies equally well to any infinite set of primes satisfying certain conditions. By applying the method to the set of primes congruent to 3 (mod 4), for example, it is shown that there is an infinite number of integer pairs n, $n+4$, each having exactly one prime factor congruent to 3 (mod 4). (Received July 28, 1941.)

362. Ivan Niven: *Quadratic diophantine equations in the rational and quadratic fields.*

Consider the general quadratic equation in two variables with rational integral coefficients, with non-negative discriminant (this restriction being imposed in order that the graph of the equation be not restricted to a finite region of the plane). Then one solution in integers of this equation implies infinitely many such solutions if and only if the graph of the equation is not an hyperbola with rational asymptotes or a pair of essentially irrational straight lines. If the coefficients of the equation are integers of a real quadratic field, one solution in integers of the field implies an infinite number if and only if the equation does not represent one of the following: no locus, a point, a pair of straight lines with coefficients essentially outside the quadratic field, or an ellipse with totally negative discriminant. A similar theorem is obtained for imaginary quadratic fields, the results being similar to the rational case. The principal parts of these theorems are proved by use of criteria for the solvability of the Pell equation in rational and quadratic fields; the criteria for quadratic fields are determined with the help of a theorem of Hilbert on the units of algebraic fields. (Received July 28, 1941.)

Analysis

363. R. P. Agnew: *Analytic extension by Hausdorff methods.*

Let $\chi(t)$, $0 \leqq t \leqq 1$, be a mass function which generates a regular Hausdorff method of summability $H(\chi)$. Let the number r, which is the greatest lower bound of numbers ρ such that $\chi(t)$ is constant over $\rho \leqq t \leqq 1$, be called the *order* of $H(\chi)$. Let $\sum c_n z^n$ be a power series with a positive finite radius of convergence. Corresponding to each vertex ζ of the Mittag-Leffler star, let $B(r, \zeta)$ denote the set of points z for which $|z-(1-r^{-1})\zeta| < r^{-1}|\zeta|$. Let $B(r)$ denote the set of inner points of the intersection of the sets $B(r, \zeta)$. It is shown that $\sum c_n z^n$ is uniformly summable $H(\chi)$ over each closed subset F of $B(r)$. The geometric series $\sum z^n$ is non-summable $H(\chi)$ at each point exterior to the closure of $B(r)$. A series $\sum u_n$ is called summable $\mathcal{H}$ to σ if there is at least one regular method $H(\chi)$ which evaluates $\sum u_n$ to σ. Some properties of the method $\mathcal{H}$ are obtained. The existence of series with bounded partial sums which are not summable $\mathcal{H}$ is implied by the following Tauberian gap theorem. If $0 < n_1 < n_2 < \cdots$, if $n_{p+1}/n_p \to \infty$ as $p \to \infty$, if $u_n = 0$ when $n \neq n_1, n_2, \cdots$, and if $\sum u_n$ is summable $\mathcal{H}$, then $\sum u_n$ is convergent. (Received June 23, 1941.)

364. R. P. Agnew: *On Hurwitz-Silverman-Hausdorff methods of summability.*

Let $H(\chi)$ denote a regular transformation of the type defined by Hurwitz and Silverman and by Hausdorff, and let $\mu(z)$ be the moment function generated by $\chi(t)$. It is shown that if q is a complex number for which $q\neq 0$ and $\mathcal{R}q\geqq 0$, then the sequence $k!/(k-q)!$ is summable $H(\chi)$ if and only if $\mu(q)=0$. Applications are given to the problems of relative inclusion of methods $H(\chi)$, inclusion of methods $H(\chi)$ by Abel's method, and omission and adjunction of terms in summable series. (Received July 25, 1941.)

365. R. P. Agnew and Mark Kac: *On differences of elements of sets having positive Haar measure.*

Let $|A|$ denote the Haar measure of a set A in a metric separable locally compact space E whose elements x constitute a continuous group. Using $+$ for the symbol of combination, let the element 0 for which $x+0=0+x=x$ be the origin of the space, and let $-x$ denote the inverse of x. It is shown that if $|A|>0$, then the set $\Delta(A)$ of points x representable in the form $a-b$ where $a, b\in A$ contains an open set containing the origin. Applications and related results are given. (Received July 25, 1941.)

366. J. D. Bankier and Walter Leighton: *Arithmetic continued fractions.*

Galois' theorem is generalized. A convergent pure periodic continued fraction, none of whose partial numerators is equal to zero, is considered. The value y of the continued fraction is determined by solving a quadratic equation, and the other root z is called the conjugate of y. It is shown that if the continued fraction which is formed by reversing the period converges, its value is $-z$. This result is used to study proper continued fractions. It is proved that, if $y=w+I$, where w is a reduced quadratic irrational with an odd number of terms in the period of its regular continued fraction expansion, and I is an integer, then y admits an infinitude of periodic proper continued fraction expansions, the partial numerators being equal to the same positive integer and the period consisting of two terms with a preliminary period of two terms. Necessary and sufficient conditions that a fraction be equal to an approximant of a proper continued fraction are obtained. (Received July 28, 1941.)

367. E. F. Beckenbach: *Vector formulations of Morera's theorem.*

Integral conditions are given which imply that a continuous vector of components $X_i(x_1, x_2, \cdots, x_n), i=1, 2, \cdots, n$, be both irrotational and solenoidal. For $n=2$, the results reduce to Morera's theorem. (Received August 1, 1941.)

368. Stefan Bergman and D. C. Spencer: *Some properties of pseudo-conformal transformations in the neighborhood of boundary points.*

In this paper the theorem of Lindelöf on the behavior of an analytic function in the neighborhood of a discontinuity is generalized to a pair of functions of two complex variables. (Received July 28, 1941.)

369. D. G. Bourgin: *Some properties of Banach spaces.*

The interrelations of various characterizations of Banach spaces and their subsets are considered. The main concepts are those of the weak topologies, weak compactness and completeness of order $\aleph_a$, a generalization of quasi-uniformity of convergence, weak convergences, an extension of the Helly property to non-denumerable sets of equations and reflexivity. An example is given of a bicompact set in the unit sphere of the conjugate space, E^*, using the weak topology of E^* as functionals, which contain no non-trivial weakly convergent sequences. (Received July 28, 1941.)

370. R. H. Cameron: *Quadratic convolution equations.*

This paper deals with the second degree Stieltjes integral equation $\int\int A(x-t-u)df(t)df(u)+2\int B(x-t)df(t)+C(x)=0$, in which $A(x)$, $B(x)$, $C(x)$ are given functions of bounded variation on the infinite interval, and seeks solutions $f(x)$ of the same sort. (All integrals and sums are from $-\infty$ to ∞.) Sufficient conditions are given for the existence of two and only two normalized solutions, and also for the reality of the solutions. The special cases $\int\int P(x-t-u)\phi(t)\phi(u)dtdu+2\int[p_0\phi(t)+Q(t)]\phi(x-t)dt+2q_0\phi(x)+R(x)=0$ and $\sum\sum a_{k-\mu-\nu}u_\mu u_\nu+2\sum b_{k-\mu}u_\mu+c_k=0$ are also considered, functions and sequences being of class L_1 and l_1 respectively. (Received July 24, 1941.)

371. C. C. Camp: *A convergence proof involving an inseparable multiple contour integral.*

In removing the restriction assumed in a former paper (American Journal of Mathematics, vol. 60, p. 452), that $a_{ji}(x_j)$ maintain its average value over every subinterval for x_j involved in the auxiliary condition the author treats the analogue of $W_j(\nu_j)$, which is no longer a function of one complex variable. It is proved, however, that the real parts of the zero-places of $W_j(z_1, z_2)$ for the case $p=2$, $(j=1, 2)$, are bounded. The residues of the corresponding Green's system are then worked out according to Poincaré's forms for a simple place (z_1^*, z_2^*) and when W_j has a double root while $W_{3-j}\neq 0$ at the place $(z_1^{(\rho)}, z_2^{(\rho)})$. The proof of convergence requires two lemmas different from previous ones in that they cannot be written as products of integrals of single variables. The polycylinder interpretation is sufficient here although one might include the more general surfaces recently treated by S. Bergman. Extensions to the case of $p>2$ and to functions $a_{ji}(x_j)$ which may change sign are indicated. (Received July 18, 1941.)

372. M. M. Day: *Ergodic theorems for abelian semi-groups.*

Improving the methods of an earlier paper (abstract 46-5-257), this paper uses the second conjugate space and its compactness properties to simplify the proof of the ergodic theorem for a bounded abelian semi-group of transformations of a Banach space into itself. Various supplementary results are also given. For example there is a non-trivial invariant set-function defined over all the subsets of each abelian semi-group. This and the main theorem imply that every element of a reflexive space B is ergodic in the sense of Alaoglu and Birkhoff (Annals of Mathematics, (2), vol. 41 (1940), pp. 293–309) and under each bounded abelian semi-group of transformations in B. (Received June 28, 1941.)

373. M. M. Day: *Operations in Banach spaces.*

A theorem of Banach (*Théorie des Opérations Linéaires*, p. 80, Theorem 5) states that if A and B are Banach spaces and if U_n, $n=1, 2, \cdots$, are linear operators on A

to B such that $\limsup_n \|U_n(a)\| < \infty$ for each a in A, then $\limsup_n \|U_n\| < \infty$. This relation need not hold if the sequence of integers is replaced by any directed set X and this paper is mainly concerned with the problem of boundedness suggested by this. Characterize the triples $\{A, B, X\}$, where A and B are Banach spaces and X is a directed set, such that the operators U_x on A to B can be chosen so that $\limsup_x \|U_x(a)\| < \infty$ for every a in A while $\limsup_x \|U_x\| = \infty$. Let P be the class of such triples. Typical results are these: (1) $\{A, B, X\} \in P$ for every B if there is a B_0 such that $\{A, B_0, X\} \in P$. (2) If $\{A, B, X\} \in P$ and $Y > X$ in the sense of Tukey (*Convergence and Uniformity in Topology*, Princeton, 1940), then $\{A, B, X\} \in P$. (3) If $\{A, B, X\} \in P$ and there is a linear operator taking a Banach space A' into all of A, then $\{A', B, X\} \in P$. (4) A is not finite dimensional if and only if X and B exist so that $\{A, B, X\} \in P$. (5) X is essentially sequential if and only if no A and B exist such that $\{A, B, X\} \in P$. (Received June 28, 1941.)

374. M. M. Day: *Reflexivity criteria for a Banach space.*

An elementary proof not involving integration is given for a theorem of Goldstine (Duke Mathematical Journal, vol. 4 (1938)) that weak completeness of a general sort implies reflexivity of a Banach space B. The principal tool is a lemma on the distance from the zero element of B to the intersection of a finite number of hyperplanes in B; an attempt to use the same criterion for the intersection of $\aleph_\lambda$ hyperplanes ($\lambda \geqq 0$) leads to a new criterion for reflexivity of B and to the extension of a number of known conditions for reflexivity of B to conditions for reflexivity of every $\aleph_\lambda$-separable subspace of B. Many of the results are related to those of Šmulian (Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS, vol. 18 (1938)). (Received July 23, 1941.)

375. R. J. Duffin and J. J. Eachus: *A Paley-Wiener type expansion theorem.*

The following expansion theorem is easy to prove: Let $\{A_{nk}\}$ $(n, k = 1, 2, \cdots)$ be a matrix of constants such that $|A_{nk}| < a_k$. Let $\{T_k\}$ be a sequence of bounded linear transformations with corresponding bounds $\{t_k\}$. Let $\{f_n\}$ be a complete orthonormal sequence of functions and define $g_n = f_n + \sum_{k=1}^{\infty} A_{nk} T_k f_n$. Then if $\sum_1^\infty a_k t_k < 1$ the sequence of functions $\{g_n\}$ is strongly complete in the sense of Paley and Wiener, *Fourier Transforms in the Complex Domain*, p. 100. The modus operandi of the proof is "the method of separation of variables." The theorem is easy to apply upon observing that multiplication by a bounded function is a bounded linear transformation of Hilbert space. For example, new results are obtained for the non-harmonic Fourier series. Thus the sequence of functions $\{e^{i\lambda_n x}\}$, $n = 0, \pm 1, \cdots$ where $\{\lambda_n\}$ is a sequence of complex numbers satisfying $|\lambda_n - n| \leqq L$ is strongly complete for $L < \log 2/\pi$. This is clearly a better value than $1/\pi^2$ obtained by Paley and Wiener. The best value is not known but a theorem of Levinson gives an upper limit of $1/4$. (Received July 30, 1941.)

376. R. J. Duffin and A. C. Schaeffer: *Power series with bounded coefficients.*

The following theorem of Szegö is well known: If $f(z) = \sum_0^\infty a_n z^n$ has only a finite number of different coefficients and if $f(z)$ is continuable beyond the unit circle then it is a rational function. In this paper the following stronger assertion is proved: If $f(z)=\sum_0^\infty a_n z^n$ has only a finite number of different coefficients and if $f(z)$ is bounded in some arc of the unit circle, then it is a rational function. Of central importance in the proof is a recent result of Cartwright which states that if $\phi(\omega)$ is an entire function such that $|\phi(\omega)|=O(e^{k|\omega|})$, $k<\pi$, and if $\phi(\omega)$ is bounded for integer values of ω then it is bounded on the entire real axis. Some generalizations of Cartwright's theorem are obtained, and it is shown that these generalizations can be used to obtain gap theorems which overlap those of Paley and Wiener, *Fourier Transforms*, p. 124. (Received July 30, 1941.)

377. J. M. Earl: *Polynomials of interpolation on an infinite interval.*

If $f(x)$ is an arbitrary function of x with upper bound M, then the interpolating polynomial of degree n which assumes the value $f(x)$ for $n+1$ values of x has an upper bound which depends on M and the distribution of these $n+1$ values of x. If the interval on which $f(x)$ is defined is infinite and the manner in which $f(x)$ becomes infinite when x becomes large numerically is suitably restricted, then the distribution of the $n+1$ values of x may be chosen so that an upper bound for the magnitude of the product of a weight function and the interpolating polynomial may be obtained for the whole infinite interval. It is possible in this manner to obtain theorems on the convergence of polynomials of interpolation to a given function on an infinite interval. For example, if $f(x)$ has a pth derivative which satisfies a Lipschitz condition on the doubly infinite interval and the $n+1$ points of interpolation are equally spaced on an interval for which $|x|\leqq n^{3/4}$, then the difference of $f(x)$ and the interpolating polynomial, multiplied by the exponential of the negative of x square, does not exceed a constant multiple of $n^q \log n$, where $q=-(p+1)/4$. (Received July 31, 1941.)

378. Paul Erdös and Gabor Szegö: *On a problem of I. Schur.*

Let $n\geqq 3$, $-1\leqq x_0\leqq +1$, and let $K_n(x_0)$ denote the class of polynomials $f(x)$ of degree n for which $|f(x)|\leqq 1$ in $-1\leqq x\leqq +1$ and $f''(x_0)=0$. The main purpose of this paper is the determination of the maximum $m_n\cdot n^2$ of $|f'(x_0)|$ as $f(x)\in K_n(x_0)$ and x_0 runs from -1 to $+1$. It is shown that, for sufficiently large n, this maximum is attained for $x_0=\pm 1$ and for certain polynomials which were introduced and investigated by G. Zolotareff. Furthermore $\lim_{n\to\infty} m_n = k^{-2}(1-E/K)^2=0.3124\cdots$ where k^2 is the only root of the transcendental equation $(K-E)^3+(1-k^2)K-(1+k^2)E=0$, $-1<k^2<+1$. Here K and E denote the complete elliptic integrals associated with the modulus k^2. A brief study of the polynomials of Zolotareff is included. Short proofs of two theorems of Zolotareff are given. (Received August 1, 1941.)

379. L. R. Ford: *The proper fractions.*

The proper fractions are the rational numbers between 0 and 1. The following formula is typical of the results obtained in this paper: $\sum(q^2 \sin p\pi/q)^{-2}=5/\pi^2-1/3$, where the summation extends over all proper fractions p/q. (Received July 28, 1941.)

380. Bernard Friedman: *Fourier coefficients of odd functions.*

Given an odd measurable function $f(x)$ whose absolute value is less than one. Let $\sum b_n \sin nx$ be its Fourier series. Then an nth order determinant is obtained which expresses the necessary and sufficient condition satisfied by the first n coefficients. The method is similar to that of a previous paper (this Bulletin, vol. 47 (1941), pp. 84–92). (Received July 30, 1941.)

381. H. L. Garabedian: *A class of linear integral transformations.*

This paper involves a study of a class of integral (kernel) transformations, first considered by Silverman, which has its analogue in the Hausdorff theory of matrix transformations. In view of certain recent developments in the field of Hausdorff matrix transformations it has been found possible to clarify certain obscurities in Silverman's work, and to make significant extensions of Silverman's results on inclusion and equivalence relations among Hausdorff integral transformations. (Received July 3, 1941.)

382. H. L. Garabedian: *Theorems relating to the Cesàro kernel transformation.*

It is the object of this paper to determine conditions on the kernel $k(s, t)$, of a regular transformation $z(s)=\int_0^s k(s, t)x(t)dt$, in order that the method of summation defined by the transformation shall include Cesàro summability (C, α) with the kernel $\alpha(1-t/s)^{\alpha-1}/s$, $\alpha>0$. Two theorems are obtained, corresponding to the two cases in which α is an integer or is not an integer, and examples are given in illustration of them. (Received July 3, 1941.)

383. Abe Gelbart: *On some properties of mapping functions.*

In this paper inequalities are obtained for the coefficients of the mapping function in one complex variable which transforms a given domain, simply- or multiply-connected, into a schlicht domain which omits three points, by using the method of the minimum integral (Bergman, Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 16 (1937), pp. 11–14). Analogous inequalities can be obtained for functions of two complex variables. (Received July 28, 1941.)

384. T. F. Glass and Walter Leighton: *On the convergence of continued fractions.*

Sufficient conditions for the convergence of the continued fraction $K[a_n/1]$ are that $|a_{2n+1}| \leqq r$, where r is an arbitrary positive number $\leqq \frac{1}{4}$, and that the numbers $a_{2n}=\rho_{2n}e^{i\theta_{2n}}$ lie in a region defined by the relations: $\rho_{2n} \geqq 2(1+r)^2[1-\cos(\theta_{2n}+\alpha_0)]$ $(0 \leqq \theta_{2n} \leqq \pi-\alpha_0)$, $\rho_{2n} \geqq 2(1+r)^2$ $(\pi-\alpha_0 \leqq \theta_{2n} \leqq \pi+\alpha_0)$, $\rho_{2n} \geqq 2(1+r)^2[1-\cos(\theta_{2n}-\alpha_0)]$ $(\pi+\alpha_0 \leqq \theta_{2n} \leqq 2\pi)$, where $\alpha_0=2 \arcsin r$. This sharpens a criterion due to Leighton and Wall. (Received July 28, 1941.)

385. P. R. Halmos and John von Neumann: *Operator methods in classical mechanics.* II.

A measure space is *normal* if (1) points have measure zero, (2) the family of all measurable sets contains a Borel field B with a countable number of generators, sets of which "cover" every measurable set and separate every two points, and (3) the range of each real-valued univalent function $f(x)$, measurable (B), is a Borel set. A measure space is isomorphic to the unit interval if and only if it is normal. A subset E of the unit interval is *absolutely invariant* if for every measure preserving transformation T of the unit interval on itself the symmetric difference of E and TE has measure zero. A measure space satisfying conditions (1) and (2) is isomorphic to an absolutely invariant set if and only if every measure preserving automorphism of the Boolean algebra of measurable sets modulo sets of measure zero is induced by a point transformation. Every measure preserving transformation with discrete spectrum on a normal space is isomorphic to a translation on a compact separable abelian group, and hence to an isometric transformation. Conversely every isometric ergodic measure preserving transformation of a metric, complete, separable space with a regular measure has discrete spectrum. Similar results hold for measurable flows. (Received July 11, 1941.)

386. F. B. Jones: *Measure and other properties of a Hamel basis.*

A Hamel basis is a set of real numbers $a, b, c, \cdots$ such that each real number can be expressed uniquely in the form $\alpha a+\beta b+\gamma c+\cdots$, where $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero. Certain measure conditions under which a set (of a large class of sets, including all analytical sets) shall contain a Hamel basis are obtained which are both necessary and sufficient. Although no Hamel basis is an analytical set, an example of a Hamel basis which contains an infinite perfect set is given. Some of the rather peculiar properties of this perfect set are pointed out. (Received July 21, 1941.)

387. Mark Kac: *Convergence and divergence of non-harmonic gap series.*

Consider the series $\sum_{n=1}^{\infty} a_n e^{i\lambda_n t}$, where λ_n is a real number satisfying the gap condition $\lambda_{n+1}/\lambda_n > q > 1$ $(n=1, 2, \cdots)$. If $\sum_{n=1}^{\infty}|a_n|^2 < \infty$, the first series converges almost everywhere for $-\infty < t < \infty$. If a stronger gap condition is assumed; namely, $\lambda_{n+1}/\lambda_n > q > (5^{1/2}+1)/2$, then the divergence of $\sum_{n=1}^{\infty}|a_n|^2$ implies the divergence almost everywhere of the first series. The proofs refer to the work of Marcinkiewicz and Zygmund on harmonic gap series. (Received July 26, 1941.)

388. S. Kakutani: *A class of examples of mixing flows*

The following theorem is proved: Let Ω be the space of all sequences $\{x_n\}$ $(n=\cdots, -1, 0, 1, \cdots)$, with $0 \leqq x_n \leqq 1$, and with the usual independent measure defined multiplicatively in terms of Lebesgue measure on each coordinate. Let T be the transformation on this space taking each point $\omega=\{x_n\}$ into the point $T\omega=\{x_{n+1}\}$ and let T_t be a flow built on this transformation under a function $f(\omega)$ which depends only on a single coordinate: $f(\omega)=f(x_0)$. (For the definition of a flow built under a function see W. Ambrose, Annals of Mathematics, July, 1941.) Then T_t is a strong mixing flow unless all values taken by the function $f(\omega)$ (except for a set of measure zero) are integral multiples of some constant c. If all values taken by $f(\omega)$ are integral multiples of some constant c, then the flow has a non-empty point spectrum. The proof of this theorem is accomplished by use of a Tauberian theorem of N. Wiener. (Received July 26, 1941.)

389. S. Kakutani and W. Ambrose: *Structure and continuity of measurable flows.*

Various theorems about the structure of measurable flows are obtained. In particular it is shown that on a measure space (i.e., a space on which a countably additive finite valued measure is defined) satisfying the following two conditions: (1) there exists a (countable) sequence of measurable sets such that for every pair of points some member of the sequence contains one but not both of the points, and (2) the measurable sets are obtained by completing (i.e., by throwing in all subsets of sets of measure zero) a Borel field determined by a countable collection of sets (both of these conditions are clearly satisfied by Lebesgue measure in euclidean space) every measurable flow (other than the identity) is isomorphic to a flow built under a function (for definition see Ambrose, Annals of Mathematics, July, 1941) and also to a continuous flow on a separable metric space with a regular measure. (Received July 26, 1941.)

390. R. B. Kershner: *On the packing of convex regions in the plane.* Preliminary report.

By a packing of a convex region R in the plane is meant a configuration consisting of a finite or infinite collection of congruent images of R placed in a nonoverlapping way upon the euclidean plane. The density of such a packing is defined in an obvious way and represents the proportion of space covered. By the packing constant $\gamma(R)$, for a given convex region R, is meant the least upper bound of the set of densities corresponding to all packings of R. It is shown that there is an absolute constant $k>0$ such that $\gamma(R)\geqq k$ for all convex regions R. The regions (polygons) for which $\gamma(R)=1$ are determined. As a lemma the area A_n of the minimum n-gon containing a given convex region of area A is discussed and the order of magnitude of $1-(A_n/A)$ is determined. (Received July 28, 1941.)

391. R. E. Langer: *A theory for ordinary linear differential boundary problems of highly irregular type.*

A general theory for boundary problems, which in matrix notation are $y'=\{\lambda R(x)+Q(x)\}y$, $W_a(\lambda)y(a)+W_b(\lambda)y(b)=0$, has hitherto been given only for cases classified as regular or mildly irregular. For highly irregular problems, results have been obtained only in markedly special cases (papers by J. W. Hopkins, L. E. Ward, and J. I. Vass), and even then no theorems of a generality comparable with those known for regular problems have been found. In this paper the highly irregular problem is approached by a new method, in which it is imbedded in a continuous aggregate of problems of which all other members are regular. Its expansion theory is thus sought through limiting processes from existing theory. A sub-classification of highly irregular problems is found to be called for. For the expansions associated with problems of the one category (which includes all those for which any discussions are in the literature) the convergence of regular expansions is made to yield an appropriately defined summability, under conditions familiar from the theory of Fourier's series. For problems of the other category the method proves inapplicable, and it seems probable that no expansion properties in the usual sense inhere in these cases. (Received July 26, 1941.)

392. J. D. Mancill: *Multiple integral problems of the calculus of variations with prescribed transversality coefficients.*

Single integral problems of the calculus of variations with prescribed transversality conditions have recently been studied by the author (American Journal of Mathematics, vol. 41 (1939), pp. 330–334). In the present paper it is shown that the method used in that study applies with slight modification to multiple integral problems. The method applies with equal facility to parametric and non-parametric integrals of all orders of multiplicity but the case of double integrals is presented in this paper for the sake of simplicity of notation. The problem is stated in parametric form and necessary and sufficient conditions in order that a transversality relation belong to such a problem of the calculus of variations are derived. Finally the most general integrand function of such a problem to which a given transversality relation belongs is determined. (Received June 18, 1941.)

393. Szolem Mandelbrojt: *New proof of the conditions of quasi-analyticity.*

The author gives a very simple proof for the necessity of the well known conditions that a family of functions be quasi-analytic. (Received July 28, 1941.)

394. Ralph Mansfield: *Differential systems involving k-point boundary conditions.*

This paper is concerned with a k-point boundary value problem consisting of a system of ordinary linear differential equations and a set of boundary conditions involving linearly the values of the solutions at interior points as well as at the end points of the interval over which the system is defined. By means of a transformation this boundary problem is reduced to a two-point problem. By this device the fundamental properties of the k-point problem and its adjoint system are readily deduced. The definitions of self-adjoint, definitely self-adjoint, self-conjugate adjoint, and definitely self-conjugate adjoint systems are extended to the k-point boundary problem. In addition, necessary and sufficient conditions that such k-point problems arise from certain problems of the calculus of variations are determined. Applications of this method to the study of boundary problems in the complex domain lead to some of the results previously established by Langer. (Received June 3, 1941.)

395. Herman Meyer: *Polynomial approximations to functions defined on abstract spaces.*

The purpose of this paper is to extend the Weierstrass theorem on polynomial approximations to continuous and differentiable functions to the case where the domain of the functions involved is a Banach space with a basis and the range is a general Banach space. For a continuous function the approximating sequence of polynomials (i.e., polynomials in the sense of Gateaux, Bulletin de la Société Mathématique de France, vol. 47 (1919), p. 73) converges on bounded closed sets, uniformly on compact subsets. For a differentiable function, after some further restrictions on the domain, the differentials of the polynomials also converge to the corresponding differentials of the function. Subject to still further restrictions on the domain, an explicit formula to determine the polynomials is presented. A double sequence of Banach valued polynomials defined over euclidean spaces is set up and a proper diagonal subsequence is selected. Well defined linear transformations of the domain onto certain spaces isomorphic to euclidean spaces then produce the desired approximating sequence. (Received July 25, 1941.)

396. M. G. Moore: *On functions satisfying two functional equations.*

Making a study quite similar to that of doubly periodic analytic functions, an examination is made of the class of analytic functions which satisfy two equations of the type $\sum_{\mu=1}^{N} c_\mu f(x+\alpha_\mu)=0$, where c_μ and α_μ are complex constants, or, more generally, which satisfy two differential equations of infinite order of certain character. In particular, two equations cannot in general have a common non-trivial solution analytic throughout certain convex regions, although they do have solutions analytic throughout smaller regions. The complete analytic solution of a pair of equations is characterized as is also that of three equations. (Received July 29, 1941.)

397. E. H. Nicholson: *On the degree of approximation in some convergence theorems concerning derivatives of the mapping function in conformal mapping.*

Suppose that the function $w=f(z)$ maps the unit circle conformally on the interior of a closed Jordan curve C. It is known that, under suitable assumptions concerning C, the mapping function $f(z)$ and its derivatives vary continuously in the closed unit circle under a continuous deformation of C. For certain problems it is desirable to know the degree of the variation of $f(z)$ and its derivatives in this dependence. This means: if C_1 and C_2 are closed Jordan curves which "differ" by less than some positive ϵ, and if $f_i(z)$ are the corresponding mapping functions $(f_1(0)=f_2(0),\ f_i'(0)>0)$, $i=1, 2$, what can be asserted regarding an upper bound for $|f_1(z)-f_2(z)|$, $|f_1'(z)-f_2'(z)|, \cdots$ in terms of ϵ? The problem concerning the degree of variation of the function $f(z)$ itself has been investigated by L. Bieberbach, A. R. Marchenko, and A. Markouchevitch. The present paper deals with the corresponding problem for the derivatives of $f(z)$. (Received July 30, 1941.)

398. I. E. Perlin: *A calculus of variations problem with end points as functions of the curve.* Preliminary report.

In this paper the author considers a calculus of variations problem with end conditions functions of the curve. By introducing new variables the author transforms this problem to a Lagrange problem with variable end points. Through the use of a generalized Lindeberg theorem sufficient conditions for the original problem can be shown to be deduced from sufficient conditions for the related Lagrange problem (Received July 30, 1941.)

399. R. S. Phillips: *On weakly compact subsets of a Banach space.* Preliminary report.

A subset G of a Banach space X is called $\aleph_\alpha$-closed if for every subset $[x_\pi]\subset G$ defined on a directed set of cardinal power less than or equal to $\aleph_\alpha$ there exists an $x_0\in X$ such that $\lim\inf_\pi \bar{x}(x_\pi)\leqq \bar{x}(x_0)\leqq \lim\sup_\pi \bar{x}(x_\pi)$ for every $\bar{x}\in\overline{X}$, the conjugate space. $\aleph_0$-closed sets are the usual weakly conditionally compact sets, and some properties are presented. If the unit sphere of X is $\aleph_\alpha$-closed, the unit sphere of $\overline{X}$ is also $\aleph_\alpha$-closed. If in addition there exists a set of power $\aleph_\alpha$ total on X, then X is reflexive. This is equivalent to a recent result of Bourgin. A linear transformation on X to Y is called $\aleph_\alpha$-closed if the image of the unit sphere in X is $\aleph_\alpha$-closed and there exists a set of power $\aleph_\alpha$ total on the closed linear extension of this image. The adjoint transformation of a $\aleph_\alpha$-closed transformation is likewise $\aleph_\alpha$-closed. All weakly completely continuous transformations on L to X are separable valued. Finally, any completely additive set function of strong bounded variation on a sigma field to an "almost" $\aleph_0$-closed subset of X is the indefinite integral of a Bochner integrable function. (Received July 29, 1941.)

400. Harry Pollard: *Legendre polynomials and the Hausdorff moment problem.*

Among the proofs given by Hausdorff of his classical result on the representation of completely monotonic sequences there is one which makes use of the $(C, 2)$ kernels for Legendre series. In the present paper the theorem in question is deduced from the fact that the Abel kernel for these series is also non-negative. In terms of the Legendre polynomials, a new criterion is then obtained for the representation of a sequence in the form, $\mu_n=\int_0^1 t^n\phi(t)dt$, with $\phi(t)$ bounded. This yields a simple proof of Steinhaus' result on the form of the general linear functional on the space L. (Received July 31, 1941.)

401. Arthur Rosenthal: *On differentiation of integrals and approximate continuity.*

According to Lebesgue the derivative of an integral is equal to the integrand almost everywhere, and according to Denjoy the integrand (assumed as bounded) is furnished by differentiation certainly wherever it is approximately continuous. While in the generalization of Lebesgue's theorem to more dimensions or to abstract spaces, the system of sets relative to which one differentiates must be rather special, the generalization of Denjoy's theorem can be made by use of quite arbitrary indefinitely fine systems of sets. Under certain conditions the approximate continuity is shown to be necessary and sufficient for differentiation of the integral to furnish the integrand. (Received July 11, 1941.)

402. Raphael Salem: *On sets of multiplicity for trigonometrical series.*

Let P be a symmetrical perfect set constructed on $(0, 2\pi)$ by consecutive trisections, the central intervals being removed, and each of the 2^{p-1} trisections of the pth step being made proportionally to ξ_p, $1-2\xi_p$, ξ_p $(0<\xi_p\leqq\frac{1}{2}$; $p=1, 2, \cdots)$. Let δ_{pk} $(k=1, 2, \cdots, 2^p-1)$ be the 2^p-1 intervals removed after the pth step and let $F(x)$ be the singular monotonic function of the Cantor type constructed on P and such that $F(x)=k/2^p$ in δ_{pk}. Let c_n be the Fourier-Stieltjes coefficient of dF in respect of e^{nix}. Let $\{a_k\}$, $\{b_k\}$ be two sequences such that $0<a_k<b_k<\frac{1}{2}$, and consider all the sets P for which $a_k\leqq\xi_k\leqq b_k$. Following Steinhaus, the infinitely many dimensional domain $a_k\leqq\xi_k\leqq b_k$ $(k=1, 2, \cdots)$ is mapped on the interval $0\leqq t\leqq 1$. Supposing that $b_k-a_k\geqq 1/\omega(k)$, $\omega(k)$ being increasing and such that $\log\,\omega(k)=o(k)$, the following theorems are proved: If $\lim\inf\,(a_1a_2, \cdots, a_p)^{1/p}=\alpha>0$, then for almost all t, (a) the sets P are sets of multiplicity, (b) $\sum|c_n|^s<\infty$ for $s>s_0=s_0(\alpha)$, (c) $c_n=o(n^{-\delta})$ for $\delta<\delta_0=\delta_0(\alpha)$. The result (a) holds on the more general assumption $\lim\inf\,p^{1/2-\beta}(a_1a_2, \cdots, a_p)^{1/p}>0$, β being any positive number. (Received July 8, 1941.)

403. D. C. Spencer: *A function-theoretic identity.*

Let $n(r, a)$ be the number of roots of $f=a$ in $|z|<1$. Suppose that $g(R)$, defined for $R\geqq 0$, is absolutely continuous, and that $G(R)=\int g(R)d(lnR)$. Then for any f regular in $|z|<1$, we have $r\{d/dr\}\int_{-\pi}^{\pi}G(|f|)d\phi=\int_0^r\int_{-\pi}^{\pi}g'(|f|)\cdot|f|^{-1}\cdot|f'|^2\rho d\rho d\phi+2\pi n(r,0)\cdot g(0)$. Taking $g(R)=1$, $\lambda R^\lambda(\lambda>0)$, and $1/(1+R^2)$, we obtain respectively the identities (heretofore considered distinct) of Jensen, Hardy-Stein, and Ahlfors-Shimizu (the latter expressing Nevanlinna's characteristic function $T(r)$ in terms of area on the Riemann sphere). Application is made of the Ahlfors-Shimizu formula to obtain a localization of the well known Nevanlinna theorem that, if $T(r)=0(1)$, then f has non-tangential limits almost everywhere on $z=1$. (Received July 17, 1941.)

404. Otto Szász: *Some new summability methods with applications.*

One associates with a *series to function* transform (method of convergence factors), or with a *sequence to function* transform certain triangular matrices, and considers the relations between the corresponding summability methods. Such methods permit interesting applications to Fourier series, in particular in connection with Poisson's and Lebesgue's methods of summability. (Received July 23, 1941.)

405. S. M. Ulam: *On measures for subsets of sets of measure zero.*

Given a class K of sets Z of Lebesgue measure zero, one desires to introduce a (finitely additive) measure $m_Z(X)$ for a class of subsets X of each Z with the following properties: $m_Z(Z)=1$; $m_{Z+Z'}(X)=m_{Z+Z'}(Z)\cdot m_Z(X)+m_{Z+Z'}(Z')\cdot m_{Z'}(X)$. A construction of such a measure is possible for various classes K of sets Z, in particular for the class of all Borel sets Z of Lebesgue measure zero. This problem is related to the problem of relativization in the theory of probabilities. (Received July 30, 1941.)

406. S. M. Ulam: *Theory of the operation of products of sets.* I. Preliminary report.

The product of two sets A and B is the set $A\times B$ consisting of all couples of elements (a, b), where $a\in A$, $b\in B$; $A^2=A\times A$, $A^n=A\times A^{n-1}$. A systematic investigation of the combinatorial and set-theoretical properties of this operation is undertaken. Systems of sets, closed under the boolean operations and the operation of product, are characterized abstractly. The Borel field over the class of all sets of the form $X\times Y$, where $X\in A$ and $Y\in B$ is studied. The operation a of Suslin on sets of this class is also studied. Among other results the existence of sets of all Borel classes, of analytic sets, of non-analytic sets in the above sense, is established. Two subsets E and F of A^n are called product-isomorphic if there exists a one-to-one transformation T of A into itself such that the transformation $(a_1, \cdots, a_n)\rightarrow(T(a\ ), \cdots, T(a_n)\ )$ of A^n into itself carries the set E into F. The set-theoretical properties of this notion, of importance in applications to topological and algebraical problems, are investigated. A notion of "constructiveness" for abstract algebraical structures is introduced. (For the first report on this subject, see this Bulletin, abstract 44-3-132.) (Received July 30, 1941.)

407. František Wolf: *On generalized boundary values and the uniqueness of corresponding Dirichlet problems.*

If $u(z)$ is harmonic in a domain D, $u^D(\zeta)$ is defined at a frontier point ζ as a many-valued function which takes all the limiting values of $u(z_k)$ for all sequences $z_k\subset D$ and $z_k\rightarrow\zeta$. Similarly $u_D(\zeta)$ takes only those limiting values of $u(z_k)$ for which all z_k lie in a sector of ζ. If a system of curves $L(\zeta)$ is defined such that to every frontier point ζ there is an $L(\zeta)$ lying in the sector of ζ with one endpoint at ζ, then $u_L(\zeta)$ takes all the limiting values of $u(z)$ as $z\rightarrow\zeta$ along $L(\zeta)$. If $u^D(\zeta)=0$ at all frontier points, then $u(z)$ is continuous in the closed D, and as it is harmonic $u(z)\equiv 0$. If D is the unit circle and $u_D(\zeta)=0$ at all frontier points and $|u|\leqq\exp\{1/(1-r)^m\}$, m arbitrary, then $u(z)\equiv 0$. If D is the unit circle, $L(\zeta)$ the radii, $u_L(\zeta)=0$ at all frontier points, $|u|\leqq\phi/(1-r)^m$, ϕ and m arbitrary, then $u^D(\zeta)=0$ at all frontier points except for a reducible set of them. The function $u(z)$ is harmonic in the whole plane with the exception of these points. These results hold for more general domains and more general $L(\zeta)$ (F. Wolf, Acta Mathematica, vol. 74 (1941), pp. 65–100). (Received June 12, 1941.)

408. František Wolf: *On the summability of trigonometrical integrals.*

For (C, k) sums of a trigonometrical integral $C_k(x, \omega)=\int_0^\omega(1-\lambda/\omega)^k(\cos\lambda x\, dA(\lambda)$ $+\sin\lambda x\, dB(\lambda))$ which are formally of Stieltjes type, but may be defined by one partial integration, the following results are valid: (A) If (i) $\int_0^\infty(\cos\lambda x\, dA(\lambda)$ $+\sin\lambda x\, dB(\lambda))/\lambda^{4m}=F(x)$ converges uniformly, (ii) $F(x)=(\int^x dx)^{4m}f(x)$ for $x\subset(a, b)$,

(iii) $\int_0^\omega(1-\lambda/\omega)^k dA(\lambda)$, $\int_0^\omega(1-\lambda/\omega)^k dB(\lambda)$ converge as $\omega\rightarrow\infty$, then $C_k(x, \omega)$ equiconverges with the (C, k) means of the Fourier series of $f(x)$ in $(a+\epsilon, b-\epsilon)$, $\epsilon>0$. At the same time the allied trigonometrical integral $\overline{C}_k(x, \omega)$ equiconverges with the trigonometrical series allied with the Fourier series of $f(x)$ in the same interval. (B) If $\limsup C_k(x, \omega)<\infty$ for $\omega\rightarrow\infty$ in a set of positive measure, then $C_k(x, \omega)$ and $\overline{C}_k(x, \omega)$ are bounded for almost all x of E as functions of ω and if furthermore (i) and (ii) are satisfied, then the result of the preceding theorem holds good. (C) If $|C_m(x, \omega)| \leqq \psi(x) \subset L$ in (a, b), then the result of the first theorem holds for any $k>m$. (D) If $\int_a^b |C_m(x, \omega)|^p dx<M$, then the result of the first theorem holds for any $k>m+1/p$. (Received June 23, 1941.)

409. J. W. T. Youngs: *A generalized Lebesgue integral.*

S. Banach has extended the Lebesgue integral to all bounded functions defined on a finite interval (*Théorie des Opérations Linéaires*, Warsaw, 1932). The extended integral enjoys several of the standard properties of the Lebesgue integral, but properties concerned with termwise integrability of a sequence are lacking. It is shown here that if convergence of a sequence of functions is understood to mean convergence in (a general) measure, then the Lemma of Fatou is true; that is, the extended integral is a lower semi-continuous functional on non-negative functions. This fact is used to extend further the integral to unbounded functions. All the properties mentioned above together with the usual theorems on termwise integrability of a sequence hold. Several applications are made. (Received July 24, 1941.)

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410. E. L. Buell: *On the distribution of plane stress in a semi-infinite plate with partially stiffened edge.* Preliminary report.

The concentration of stress at the straight edge of a thin semi-infinite plate near the point of application of a concentrated shear load acting in the plane of the plate is reduced if the load is applied, not directly to the plate, but to an elastic stiffening rod attached along its edge. When this rod does not extend along the entire edge the boundary value problem of bi-potential theory for the Airy stress function has non-uniform boundary conditions. To solve this problem a conformal mapping of the slit full plane into the interior of the unit circle is employed. A solution of the resulting transformed boundary value problem is obtained in the form of a Fourier series, the coefficients of which satisfy an infinite system of linear equations in an infinite number of unknowns. This system has been solved approximately for the case of a stiffening rod extending to infinity in one direction from the loading point, care being taken to improve the convergence by first separating out the discontinuous parts of the solution. The resultant expressions for those stresses which are of interest have been derived and evaluated numerically. (Received July 28, 1941.)

411. A. S. Householder: *A theory of steady-state activity in nerve-fiber networks.* IV: *n circuits with a common synapse.*

Let A_i be the product of the activity parameters for the ith circuit (for terminology see Bulletin of Mathematical Biophysics, vol. 3 (1941), pp. 63–69, 105–112). Then an arbitrary stimulus pattern (SP) determines uniquely an activity pattern (AP) if and only if every A_i, as well as the sum of any number of distinct A_i, is less than unity. In case this condition fails only for the sum of all the A_i, then the possible