ON THE MAPPING OF THE SETS OF 24 POINTS OF THE SYMMETRIC SUBSTITUTION GROUP G_{24} IN ORDINARY SPACE UPON A HYPERQUADRIC CONE

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Introduction. The mapping of the sextuples of the symmetric substitution group G_6 in a plane upon a quadric has been done by Emch.[1] The 24 permutations of 4 elements x_1, x_2, x_3, x_4 considered as projective coordinates in ordinary space determine a configuration[2] which may be mapped on a hypersurface in S_4. I shall show that the hypersurface on which we will map is a hyperquadric cone. The map of every configuration on the hyperquadric will be a configuration in ordinary space, invariant under the G_{24}.

The mapping of the G_{24}. We shall represent the elementary symmetric functions as follows:

$$\begin{aligned} \phi_1 &= x_1 + x_2 + x_3 + x_4, \\ \phi_2 &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4, \\ \phi_3 &= x_1x_2x_3 + x_1x_3x_4 + x_1x_2x_4 + x_2x_3x_4, \\ \phi_4 &= x_1x_2x_3x_4. \end{aligned}$$

Let $y_i = A_i\phi_1^4 + B_i\phi_1^2\phi_2 + C_i\phi_2^2 + D_i\phi_1\phi_3 + E_i\phi_4$ where $i = 1, 2, 3, 4, 5$. There are five linearly independent y's. We shall consider the y's as the coordinates of a point in S_4. Thus to each point in (x), and consequently to each of 24 points in (x), corresponds a point (y) in S_4. The locus of the points (y) is a hypersurface of some order in S_4.

Let us choose five linearly independent y's. (For every choice of y's we will get some hypersurface and all these hypersurfaces will be linearly related.)

$$\begin{aligned} \rho y_1 &= \sum x_1^4 = \phi_1^4 - 4\phi_1^2\phi_2 + 2\phi_2^2 + 4\phi_1\phi_3 - 4\phi_4, \\ \rho y_2 &= \sum x_1^2x_2^2 = \phi_2^2 - 2\phi_1\phi_3 + 2\phi_4, \\ \rho y_3 &= \sum x_1^3x_2 = \phi_1^2\phi_2 - 2\phi_2^2 - \phi_1\phi_3 + 4\phi_4, \\ \rho y_4 &= \sum x_1^2x_2x_3 = \phi_1\phi_3 - 4\phi_4, \\ \rho y_5 &= \sum x_1x_2x_3x_4 = \phi_4. \end{aligned}$$

If we eliminate the ϕ's we get a hyperquadric cone Q given by

[1] This Bulletin, vol. 33 (1927), pp. 745–750.

[2] Veronese Annali di Mathematica, (2), vol. 2, p. 93.

$$(1)\quad y_1(y_2 + 2y_4 + 6y_5) + 2y_2^2 - y_3^2 - y_4^2 + 4y_2y_4 + 12y_2y_5 - 2y_3y_4 = 0.$$

The rank of the matrix of this hyperquadric cone is three. This means that the hyperquadric has a line of vertices. The partial derivatives,

$$\frac{\partial Q}{\partial y_1} = y_2 + 2y_4 + 6y_5, \qquad \frac{\partial Q}{\partial y_2} = y_1 + 4y_2 + 4y_4 + 12y_5,$$

$$\frac{\partial Q}{\partial y_3} = -2y_3 - 2y_4, \qquad \frac{\partial Q}{\partial y_4} = 2y_1 + 4y_2 - 2y_3 - 2y_4,$$

$$\frac{\partial Q}{\partial y_5} = 6y_1 + 12y_2$$

all vanish at the points $V(-4, 2, 4, -4, 1)$ and $V'(4, -2, -1, 1, 0)$ and any point on the join of these two points. Hence this join VV' is the vertex of the hyperquadric cone.

Next, the exceptional points of the (1, 24) transformation will be considered. To the intersections of $\phi_1=0$, $\phi_2=0$, $\phi_4=0$, that is, $(1, \omega, \omega^2, 0)$, $(1, \omega, 0, \omega^2)$, $(1, 0, \omega, \omega^2)$, $(0, 1, \omega, \omega^2)$, $(1, \omega^2, \omega, 0)$, $(1, \omega^2, 0, \omega)$, $(1, 0, \omega^2, \omega)$, $(0, 1, \omega^2, \omega)$, corresponds $y_1=y_2=y_3=y_4=y_5=0$, which represents no point. These 8 points are fundamental points of the transformation. Hereafter they will be called the F-points.

To the first neighborhood of the F-points corresponds the join of $V'(4, -2, -1, 1, 0)$ and $V(-4, 2, 4, -4, 1)$. For example, to the first neighborhood of $(1, \omega, \omega^2, 0)$, that is, $P_d(1+d_1, \omega+d_2, \omega^2+d_3, d_4)$, corresponds

$$\begin{aligned}
y_1 &= 4(d_1 + d_2 + d_3) = 4(d_1 + d_2 + d_3 + d_4) - 4(d_4),\\
y_2 &= -2(d_1 + d_2 + d_3) = -2(d_1 + d_2 + d_3 + d_4) + 2(d_4),\\
y_3 &= -(d_1 + d_2 + d_3 - 3d_4) = -1(d_1 + d_2 + d_3 + d_4) + 4(d_4),\\
y_4 &= d_1 + d_2 + d_3 - 3d_4 = 1(d_1 + d_2 + d_3 + d_4) - 4(d_4),\\
y_5 &= d_4 = 0(d_1 + d_2 + d_3 + d_4) + 1(d_4),
\end{aligned}$$

which is the join of V and V'. If P_d is on $\phi_1=0$, to it corresponds the point $V(-4, 2, 4, -4, 1)$. To any point on $\phi_1=0$, $\phi_4=0$ corresponds the point $T(2, 1, -2, 0, 0)$. A generic hyperplane, $y_1+\lambda_1y_2+\lambda_2y_3+\lambda_3y_4+\lambda_4y_5=0$, cuts Q in a quadric q and the line VT in a point R on q to which corresponds in (x) a quartic surface

$$(2)\quad \begin{aligned}\phi_1^4 + (\lambda_2 - 4)\phi_1^2\phi_2 + (4 - 2\lambda_1 - \lambda_2 + \lambda_3)\phi_1\phi_3 + (2 + \lambda_1 - 2\lambda_2)\phi_2^2\\
+ (2\lambda_1 + 4\lambda_2 - 4\lambda_3 + \lambda_4 - 4)\phi_4 = 0,\end{aligned}$$

which has $\phi_1=0$, $\phi_4=0$ (which is composed of 4 lines) as double tangents. That is, the line $\phi_1=0$, $x_4=0$ is tangent to (2) at the points $(1, \omega, \omega^2, 0)$ and $(1, \omega^2, \omega, 0)$; the line $\phi_1=0$, $x_3=0$ is tangent to (2) at the points $(1, \omega, 0, \omega^2)$ and $(1, \omega^2, 0, \omega)$; the line $\phi_1=0$, $x_2=0$ is tangent to (2) at the points $(1, 0, \omega, \omega^2)$ and $(1, 0, \omega^2, \omega)$ and the line $\phi_1=0$, $x_1=0$ is tangent to (2) at $(0, 1, \omega, \omega^2)$ and $(0, 1, \omega^2, \omega)$. Thus to a generic point R on VT corresponds the first neighborhood of the F points, on $\phi_1=0$, $\phi_4=0$.

To a hyperplane through VV'

$$y_1 + \lambda_1 y_2 + \lambda_2 y_3 + (2\lambda_1 + \lambda_2 - 4)y_4 + (6\lambda_1 - 12)y_5 = 0$$

corresponds the quartic

$$\phi_1^4 + (\lambda_2 - 4)\phi_1^2\phi_2 + (2 + \lambda_1 - 2\lambda_2)\phi_2^2 = 0,$$

which is the product of two quadrics of the form $\phi_1^2+\mu\phi_2=0$. Thus a generic hyperplane of the bundle through VV' cuts Q in two planes to which correspond in (x) two quadrics of the symmetric pencil $\phi_1^2+\mu\phi_2=0$.

To a hyperplane through VV' tangent to Q at some point $P(a, b, c, d, e)$ on Q and not on VV',

$$(b + 2d + be)y_1 + (a + 4b + 4d + 12e)y_2 - (2c + 2d)y_3 + (2a + 4b - 2c - 2d)y_4 + (6a + 12b)y_5 = 0,$$

corresponds

$$(b + 2d + be)\phi_1^4 - (4b + 10d + 2c + 24e)\phi_1^2\phi_2 + (a + 6b + 4c + 12d + 24e)\phi_2^2 = 0.$$

This quartic surface is the square of a quadric if $(4b+10d+2c+24e)^2 - 4(b+2d+be)(a+6b+4c+12d+24e)=0$ or if $-4[(2b^2-c^2-d^2) + a(b+2d+6e)+4bd+12be+2cd]=0$. But this is simply the condition that the point $P(a, b, c, d, e)$ lie on Q which we assumed in the beginning. Thus to a hyperplane through VV' tangent to Q at some point P not on VV' corresponds a quartic which is the square of a quadric $\phi_1^2+\mu\phi_2=0$.

To a hyperplane through $VV'T$,

$$y_1 + 6y_2 + 4y_3 + 12y_4 + 24y_5 = 0,$$

corresponds the quartic $\phi_1^4=0$ which is the plane $\phi_1=0$ counted four times.

To a hyperplane through the line VT,

$$y_1 + (2\lambda_2 - 2)y_2 + \lambda_2 y_3 + \lambda_3 y_4 + (4\lambda_3 - 8\lambda_2 + 8)y_5 = 0,$$

corresponds the quartic

$$\phi_1^4 + (\lambda_2 - 4)\phi_1^2\phi_2 + (8 - 5\lambda_2 + \lambda_3)\phi_1\phi_3 = 0,$$

which is composed of the plane $\phi_1=0$ and the cubic surface $\phi_1^3+(\lambda_2-4)\phi_1\phi_2+(8-5\lambda_2+\lambda_3)\phi_3=0$.

In general if a hypersurface contains a plane of Q, a factor $\phi_1^2+\mu\phi_2$ splits off of the corresponding surface in (x). And if the hypersurface contains the line VT, the factor ϕ_1 splits off in (x).

Mapping of intersections of the hyperquadric cone. A generic hypersurface H_n cuts Q in a surface F_{2n} to which corresponds in (x) a surface F'_{4n}. From the form of the transformation one can see that each of the four lines $\phi_1=0$ and $\phi_4=0$ is an n-fold double tangent, and each of the 6 points of intersection of $\phi_1=0$, $\phi_2=0$, $\phi_3=0$, $(1, i, -1, i)$, $(1, -i, -1, i)$, $(1, i, -i, -1)$, $(1, -i, i, -1)$, $(i, -i, 1, -1)$, $(i, -i, -1, 1)$, is an n-fold point of F'_{4n}.

To a generic surface F_n' in (x) corresponds on Q a surface whose order can always be determined. Suppose F_n' does not pass through the F points. The equation of F_n' will contain a term of the form ϕ_3^m where $3m=n$. A generic quartic surface F_4' and another quartic surface f_4' cuts F_n', or F'_{3m}, in $48m$ points which form $2m$ sets of 24 points each. To these $2m$ points correspond in (y) the $2m$ points that are on the plane of intersection of the two hyperplanes F and f that correspond to the two quartic surfaces F_4' and f_4'. But these $2m$ points are the intersections of the surface in (y), that corresponds to F_n', and the plane common to F and f. Thus the surface in (y) that corresponds to F_n' is of order $2m$, where $3m=n$.

The surface F_{2n} on Q is cut out by a hypersurface H which may pass through a plane of Q. For example, when F_n' is a sextic surface, H is a hyperquadric, call it H_2, which passes through a plane of Q. That is, the intersection of H_2 and Q is composed of a plane and a cubic to which corresponds in (x) a quadric and a cubic surface. More generally H_m cuts Q in a surface to which corresponds in (x) a surface of order $4m$. In order that it reduce to $3m$ it is necessary that a factor of order m split off. We have seen that the factors will be of the form ϕ_1^d and $(\phi_1^2+\mu\phi_2)^\beta$, where $d+2\beta=m$, and H_m will contain VT two times and β planes of Q. For example, if $n=9$ and $m=3$, H_m will be a cubic that contains VT and one plane of Q.

Suppose two hypersurfaces H_m and H_n cut Q. To this intersection C will correspond in (x) the intersections of two surfaces F'_{4m} and F_{4n}

which is a curve C' of order $16mn$. Thus to C_{mn} in (y) correspond in (x) C'_{16mn}.

To a generic curve C_n' in (x) which is the complete intersection of two symmetric surfaces F_r' and F_s', where $rs=n$, corresponds a curve in S_4 whose order can be determined. If the two surfaces do not go through the F points, each surface will have a term of the form ϕ_3^d where $3d=r$ or $3d\beta=s$. A surface F_4' will intersect F_r' and F_s', and consequently C_n', in $36d$ points to which corresponds in (y) $36d\beta/24$ points which are intersections of the hyperplane that corresponds to F_4' and the curve in (y) that corresponds to C_n'. Hence order of the curve in (y) that corresponds to C_n' is $36d\beta/24$ or $\frac{1}{6}n$.

The order of the curve C'_{16mn} in (x) that corresponds to the intersection of H_m and H_n on Q may be reduced if either or both of H_m and H_n contain a plane of Q or the line VT. For example if H_m contains VT then the curve in (x) is $C_{16n(m-1)}$ and if H_m contains a plane of Q the curve in (x) is $C_{16n(m-2)}$.

Symmetric quartics. To a net of hyperplanes through a line s cutting Q in A and B corresponds in (x) a net of quartic surfaces with the same 4 double tangents $\phi_1=0$, $\phi_4=0$ and with two sets of 24 points each A' and B' corresponding to A and B as base points outside of the 8 F-points which are the points of tangency. When s is tangent to Q the quartic surfaces in (x) are all tangent to each other at 24 points. Now consider any two quadrics q' and q'' on Q. The common hypertangent planes of q' and q'' envelop two hyperquadric cones. Through a generic point of Q there are two tangent hyperplanes to each of the cones. To q' and q'' correspond in (x) two quartic surfaces F_4' and F_4''. Every tangent hyperplane of one of these cones cuts Q in a quadric which touches q' and q''. To this quadric corresponds a quartic surface in (x) which touches F_4' in 24 points, and F_4'' in 24 points. That is, given two symmetric quartic surfaces F_4' and F_4'' there exist two systems of symmetric quartic surfaces such that every quartic of the system has 24 point contact with F_4' and F_4''.

To the intersection of a hyperplane through VT with Q corresponds in (x) a system of symmetric cubic surfaces $\phi_1^3+\lambda_1\phi_1\phi_2+\lambda_2\phi_3=0$. Let q' be a quadric not through VT. Now let I be the vertex of a hyperquadric cone through q' whose tangent hyperplanes cut Q in quadrics tangent to q'. To these correspond in (x) cubic surfaces and a quartic surface. Thus for a symmetric quartic surface corresponding to a generic quadric on Q there exists a system of cubic surfaces with the property of 24 point contact with the quartic surface.

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