Finally we wish to indicate that a procedure analogous to those of [4] enables us to associate with every function f, meromorphic in $\mathfrak{M}$, a characteristic function $T(r, f)$, $r<1$. Using the results of [5] and those of a work of Bers[9] as well as the theorem of this paper it is possible to show that, under certain hypotheses, $|f|$ possesses boundary values almost everywhere on $\mathfrak{F}^2$, if the $T(r, f)$ is uniformly bounded as $r \rightarrow 1$.

Massachusetts Institute of Technology

[9] The paper of Bers will appear in American Journal of Mathematics. A preliminary report of his work may be found in Comptes Rendus de l'Académie des Sciences, Paris, vol. 208 (1939), pp. 1273–1275 and 1475–1477.

MONOTONIC COLLECTIONS OF PERIPHERALLY SEPARABLE CONNECTED DOMAINS[1]

F. B. JONES

In my vain attempts to construct an example of a Moore space which is normal but not metric,[2] I have discovered a few simple and useful theorems about metric spaces which sound familiar but surprisingly do not seem to be known or in the literature. The following is such a theorem and deals with certain conditions under which a monotonic collection of domains contains a *countable* monotonic subcollection running upward through it. Application of the theorem to certain well ordered sequences is immediate.

Definitions.[3] A collection G of point sets is said to be *monotonic* provided that if g_1 and g_2 are elements of G then either g_1 contains g_2 or g_2 contains g_1. A subcollection H of a collection G of point sets is said to *run upward through* G provided that if g is an element of G there exists an element of H which contains g.

Definition. *A point set is said to be peripherally separable provided that its boundary is separable.*

Let S denote a locally connected metric space.

[1] Presented to the Society, February 22, 1941.

[2] See F. B. Jones, *Concerning normal and completely normal spaces*, this Bulletin, vol. 43 (1937), pp. 671–677.

[3] For the definition of certain terms and phrases, the reader is referred to R. L. Moore's *Foundation of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932, or to W. Sierpiński's *Introduction to General Topology*, Toronto, 1934, translated by C. C. Krieger.

THEOREM A. *If G is a monotonic collection of peripherally separable connected domains of S then some countable monotonic subcollection of G runs upward through G.*[4]

PROOF. Let H denote a well ordered subcollection of G which runs upward through G such that if h_2 of H follows h_1 of H, then h_1 is a *proper* subset of h_2. Suppose that H is uncountable. For each element h of H, let β_h denote the boundary of h. Let θ denote the infinite set of real numbers $0, 1, \frac{1}{2}, \frac{1}{3}, \cdots$, and for each point x of β_h let r_x denote the largest number of θ such that the circular region with center at x and radius equal to r_x lies in some element of H. For each n, $n=1, 2, 3, \cdots, \infty$, let M_{hn} denote the set of all points x of β_h such that $r_x=1/n$. Since β_h is separable, every subset of β_h is separable. Hence for each n, $n=1, 2, 3, \cdots, \infty$, there exists a countable subset N_{hn} of M_{hn} which is dense in M_{hn}. Since H is uncountable and, for each element h of H, $\sum_{n=1}^{n=\infty} N_{hn}$ is countable, there exists a countably infinite sequence $h_1, h_2, h_3, \cdots$ of elements of H such that for each positive integer i, h_{i+1} contains h_i together with all points y such that, for some n, the distance from y to N_{h_in} is less than $1/n$. Again since H is uncountable, some element g_1 of H contains $\sum h_i$. Let g_2 denote the first element of H following g_1 in H. Since g_2 contains a point not in g_1, g_2 contains a boundary point X of $\sum h_i$. Space being locally connected, there exists a sequence of points $x_{1n_1}, x_{2n_2}, x_{3n_3}, \cdots$ having X as a sequential limit point such that for each i, $i=1, 2, 3, \cdots$, n_i is a positive integer and x_{in_i} belongs to $N_{h_in_i}$. Obviously $n_i \to \infty$ as $i \to \infty$. But for some positive integer k, every point at a distance less than $1/k$ from X lies in g_2. Hence there exists an integer $\bar{\imath}$ such that when $i > \bar{\imath}$ every point at a distance less than $1/(k+1)$ from x_{in_i} lies in g_2. But x_{in_i} belongs to $N_{h_in_i}$, $i=1, 2, 3, \cdots$. Hence when $i > \bar{\imath}$, $1/n_i \geqq 1/(k+1)$, and hence $n_i \leqq k+1$. This is a contradiction since, as has already been pointed out, $n_i \to \infty$ as $i \to \infty$. So the assumption that H is uncountable is false.

COROLLARY. *In a locally connected metric space, every well ordered*

[4] Compare with certain of the properties discussed by Sierpiński in his paper, *Sur l'équivalence de trois propriétés des ensembles abstraits*, Fundamenta Mathematicae, vol. 2 (1921), pp. 179–188. This paper contains references to the work of Fréchet which is closely related to Theorem A. The relation of Theorem A to certain well known covering theorems (associated with the names of Borel, Lebesgue, and Lindelöf) is evident. See also R. L. Moore, *An acknowledgement*, Fundamenta Mathematicae, vol. 8 (1926), pp. 374–375; R. G. Lubben, *Concerning limiting sets in abstract spaces* II, Transactions of this Society, vol. 43 (1938), pp. 482–493; and the references therein.

increasing sequence of peripherally separable connected domains is countable.

Examples and remarks. If the hypothesis of the theorem is weakened in any respect and not strengthened in some other respect, the conclusion does not follow. This can be seen by considering the well known space which may be roughly described as composed of uncountably many straight line intervals having one common endpoint and each pair being perpendicular at that point. This example also shows (by removing $\aleph_1$ of the free endpoints one at a time) that if the word *upward* in Theorem A is changed to *downward* (and a natural interpretation given to its meaning), the resulting proposition is false. Furthermore, the theorem does not necessarily hold true for non-metric spaces, even if the space be a Moore space. The only example which I have been able to discover that shows this latter situation is unfortunately too complicated to warrant its inclusion in this paper. In still another direction, if S is metric but not locally connected, the theorem is again false. For consider a space constructed roughly in the following way. (1) Let α denote an uncountable well ordered sequence of distinct points $A_1, A_2, A_3, \cdots$ such that no point of α is preceded by uncountably many points of α. (2) For each point A_z of the sequence α, join A_z to A_{z+1} with a unit straight line interval of points such that no two such intervals have a point in common except when the end of one is the beginning of the other and preserve the ordinary limit point relations as given by these intervals (not by α). Let Q denote the space obtained so far. It consists of uncountably many mutually exclusive straight line *rays*. (3) To connect the space, a process involving an uncountable well ordered sequence of additions to Q is performed. For each point A of α having no immediate predecessor in α, select a simple sequence $B_{1A}, B_{2A}, B_{3A}, \cdots$ of points of α approaching A in α. For each positive integer i, add to Q a straight line interval T_{iA} which is $\frac{1}{4}$ unit long, which has one end at B_{iA}, and which is perpendicular to each other interval (whether added in (2) or (3)) containing B_{iA}. Let A be the sequential limit point of the end-points of the intervals $T_{1A}, T_{2A}, T_{3A}, \cdots$ which are distinct from $B_{1A}, B_{2A}, B_{3A}, \cdots$ respectively. (4) The sum of all the intervals thus put together constitutes a metric space S. For each point A_z of α, let D_z denote the sum (except for possibly the point A_z itself) of all the intervals in S containing a point of α which precedes A_z in α. The sequence $D_1, D_2, D_3, \cdots$ is a monotonic collection of connected do mains each of which has only one boundary point. Nevertheless no countable subsequence of $D_1, D_2, D_3, \cdots$ runs through it.

In view of the fact that the components of a domain in a locally connected space are themselves domains, one might suspect the following to be true: In a *connected* locally connected metric space every monotonic collection of peripherally separable domains contains a countable subcollection running upward through it. This is false as can be seen from the example of a space composed of uncountably many perpendicular intervals described above. However, the following proposition is true: *In a metric space, every monotonic collection of separable domains contains a countable subcollection running upward through it.* This follows from well known results.[5]

Applications. The application of Theorem A to the problem mentioned in the opening paragraph of this paper is more or less evident. It can also be used to establish rather easily the following known result: *A connected locally connected, locally peripherally separable, metric space is completely (perfectly) separable.*[6] The proof is direct and almost immediate.

THE UNIVERSITY OF TEXAS

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[5] See pages 300 and 301 of Alexandroff's paper, *Über die metrisation der im Kleinen kompackten topologischen Räume*, Mathematische Annalen, vol. 92 (1924), pp. 294–301, in particular.

[6] F. B. Jones, *A theorem concerning locally peripherally separable spaces*, this Bulletin, vol. 41 (1935), pp. 437–439.