BOOK REVIEWS

Théorie Analytique des Associations Biologiques. I. *Principes*. 1934, 45 pp. II. *Analyse Démographique avec Application Particulière à l'Espèce Humaine*. 1939, 149 pp. By A. J. Lotka. (Actualités Scientifiques et Industrielles, nos. 187, 780.) Paris, Hermann.

The first of these monographs is a general introduction to the second. It (the first) is primarily a non-mathematical discussion of what the author regards as two types of biological evolution:

(1) The *intra-spécifique*, that which is concerned with the changes in the aggregate of distributions associated with a given biological species, brought about by mutation, genetic variation, natural selection, etc.

(2) The *inter-spécifique*, that which deals with the changes in the numbers of individuals in each of several coexisting biological species or groups, which are brought about by different birth and death rates caused by interaction of the species with each other and with other environmental factors.

The author's main interest lies in the second type of evolution, which he discusses at some length by elaborating on the implication of chains of species regarded as food chains, and other ecological points. The problem of the variation of the numbers of individuals in a system of n species is finally formulated in a system of equations $\partial x_i/\partial t = F_i(x_1, x_2, \cdots, x_n)$ $(i=1, 2, \cdots, n)$ where x_j is the number of individuals in the jth species at time t. A formal solution is given for the case in which the F_i are approximately homogeneous linear functions of the $x_i - c_j$ in the neighborhood of $(c_1, c_2, \cdots, c_n)$ where $(c_1, c_2, \cdots, c_n)$ are the values of $x_1, x_2, x_3, \cdots, x_n$, respectively, for some "stationary state" for which the $\partial x_i/\partial t = 0$. However, this formal solution and its properties are hardly discussed at all. Special cases of solutions for the case $n=1$ and 2 are given. For the case $n=1$, the author shows that the Malthusian and the Verhulst-Pearl-Reed logistic growth functions are obtained by respectively assuming $F_1(x)$ to be linear and then quadratic in x.

In the second monograph the author concerns himself with the dynamics of human populations in which there is assumed to be no immigration or emigration. The first chapter in this monograph begins with the fundamental equation

$$N(t) = \int_0^w B(t-a)p(a)\,da \tag{1}$$

where $B(t-a)da$ is the number of individuals born within the time interval $t-a-da$, $t-a$, $p(a)$ is the proportion of individuals born at a given time who are still alive at age a, while $N(t)$ is the number of individuals present in the population at time t, and w is the maximum age attainable. w is replaced by $+\infty$ in much of the work. Lotka goes into considerable detail in dealing with this equation, deriving formulas for age distribution and average age in the population, rates of birth and mortality, survival curves, vital index, and conditions for a stationary population. Particular emphasis is placed on the Malthusian case, namely that in which $N(t)=N_0 \exp (rt)$, and $B(t) = B_0 \exp (rt)$ where r is the relative rate of increase in the population. A more general case is considered in which $B(t)$ (and hence $N(t)$) is a linear function of a function $\phi(t)$ and its derivatives. Particular attention is paid to the logistic function obtained by solving the equation $dN/dt=a_1N+a_2N^2$.

In the second chapter Lotka takes into account the effects of the variation of fertility with respect to age, by introducing a fertility function $m(a)$ as a multiplier in the integrand of (1), where $m(a)$ is the average number of daughters per mother borne by mothers in the age interval a, $a+da$. The introduction of $m(a)$ into (1) (setting $w=+\infty$) yields the integral equation

$$B(t) = \int_0^\infty B(t-a)p(a)m(a)da \tag{2}$$

which relates the number of births of daughters to the number of births of mothers. This problem and its ramifications are studied rather extensively for the Malthusian case. A formal solution of (2) for the Malthusian case of the form $B(t)=\sum_{i=1}^{\infty}Q_i \exp (r_it)$ is obtained, where the r_i are roots of the equation obtained by setting $B(t)= B_0 \exp (rt)$ in (2). Convergence and other fundamental questions which arise in connection with this solution are touched upon only lightly.

The third chapter deals with the composition of a population from the point of view of successive generations. Thus if N is the number of mothers in a beginning (zeroth) generation of age a, the surviving female offspring which are produced during the age interval da of mothers is $Np(a)m(a)da$, which is really the distribution of births of daughters in the first generation. The distribution $B_i(t)$ of births of daughters in the ith generation is therefore obtained by integrating $B_{i-1}(t-a)p(a)m(a)da$ from 0 to t, where $B_1(t)=Np(t)m(t)$. The total distribution of daughters $B(t)$ is found by summing the distribution functions for all generations, which yields an integral equation similar to (2). This same approach has been used by Lotka in his work on renewal theory.

The remaining four chapters deal with special problems such as determination of marriage rates, fertility rates and other indices and measures of natural increases in a population; determination of number of maternal, paternal and complete orphans; calculation of probabilities of extinction of a given line of descent and similar problems. The solution of the last problem, which has also been treated by R. A. Fisher, is extremely ingen'ous. A generation function $f(x)=\sum_{i=0}^{k}\pi_i x^i$ $(\sum_{i=0}^{k}\pi_i=1)$ is set up, where π_i is the probability that a male will have exactly i sons. $f(x)$ has many remarkable properties which enable the author to deal with the problem of descendance. For example $\partial f/\partial x]_{x=0}$ yields the average number of sons per male; the coefficient of x^s in $f^r(x)$ yields the probability that r males will have a total of s sons; the coefficient of x^s in $\pi_r f^r(x)$ which is a general term in $f(f(x))=f_2(x)$ say, is the probability that a man will have r sons and s grandsons. Similar interpretations can be placed on coefficients of x in the iterated function

$$f\{f(f[\cdots])\} = f_n(x).$$

The second monograph is well illustrated throughout by applications of the theory to actual census data taken from the United States, England, France, Germany, and several other countries. Numerous charts and tables are given for comparing theory with facts. The monograph is an excellent account of results which have been obtained during the past quarter of a century in the theory of population dynamics. Most of the results are due to the author himself. Lotka has shown a great deal of ingenuity in formulating the problems mathematically and in reaching practical solutions of the problems. Those interested in applied mathematics in the field of biology will find these monographs well worth reading.

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The Variate Difference Method. By Gerhard Tintner. (Cowles Commission for Research in Economics Monograph, no. 5). Bloomington, Indiana, Principia Press, 1940. 13+175 pp.

Various mathematical statistical methods have been proposed during recent years in attempts to describe, analyze and interpret economic time series. Regression analysis and its extension to harmonic analysis, moving averages, and the variate difference method are some of the techniques which have been used. The fundamental