A CHARACTERIZATION OF THE GROUP OF HOMOGRAPHIC TRANSFORMATIONS

P. O. BELL

1. **Introduction.** The objectives of this note are three-fold: (1) to present a new differential geometric characterization of the group of homographic transformations of a complex variable, (2) to interpret in geometrical language the significance of the invariance of the Schwarzian derivative under a homographic transformation, and (3) to characterize a general homographic transformation by its unique association with two families of concentric circles.

2. **Preliminaries.** Let the equation

$$w = w(z) \tag{1}$$

denote a conformal representation of the points $z=x+iy$ of a region R of the z-plane on the points $w=u+iv$ of a region $\overline{R}$ of the w-plane, whereby a general curve C is transformed into a curve $\overline{C}$. Let γ and $\bar{\gamma}$ denote the curvatures of C and $\overline{C}$ at corresponding points z and w, and let s and $\bar{s}$ denote corresponding lengths of arc of C and $\overline{C}$. For a given transformation (1) it is well known that the rate of variation $ds/d\bar{s}$ is a function $\lambda(x, y)$ which may be expressed in any one of the following forms $(u_x^2+u_y^2)^{-1/2}$, $(u_x^2+v_x^2)^{-1/2}$, $(u_y^2+v_y^2)^{-1/2}$, $(v_x^2+v_y^2)^{-1/2}$.

Comenetz[1] (using a different notation) has obtained, by elementary methods, the formula

$$\bar{\gamma} = \gamma\lambda + \lambda_y \cos\theta - \lambda_x \sin\theta, \tag{2}$$

wherein $\theta=\arctan(dy/dx)$, which is *the law of transformation of curvature in conformal mapping*, and the formula

$$d\bar{\gamma}/d\bar{s} = \lambda^2 d\gamma/ds + \lambda\left[\lambda_{xy}\cos 2\theta + \tfrac{1}{2}(\lambda_{yy} - \lambda_{xx})\sin 2\theta\right], \tag{3}$$

which is *the law of transformation of the rate of change of curvature with respect to arc length, under conformal mapping.*

3. **A new characterization of the group of homographic transformations.** It is known that the most general directly conformal transformation which carries circles into circles (including straight lines) is a homographic transformation. The transformation (1) will be such a transformation if and only if

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[1] Comenetz, *Kasner's invariant and trihornometry*, The American Mathematical Monthly, vol. 45 (1938), pp. 82–87.

(4) $$\lambda_{xx} \equiv \lambda_{yy}, \qquad \lambda_{xy} \equiv 0.$$

For if these relations hold, it follows from (3) that

$$d\bar{\gamma}/d\bar{s} = \lambda^2 d\gamma/ds.$$

Hence circles are transformed into circles. Conversely, if circles are transformed by (1) into circles, we must have at corresponding points of an arbitrarily selected circle C and its correspondent $\overline{C}$

$$d\bar{\gamma}/d\bar{s} = \lambda^2 d\gamma/ds = 0.$$

Equation (3) also holds. Hence, no matter what the direction of C at z may be, the following equation must hold:

(5) $$\lambda(\lambda_{yy} - \lambda_{xx})(\sin 2\theta)/2 + \lambda\lambda_{xy} \cos 2\theta = 0.$$

This equation must, therefore, be satisfied independently of θ, and conditions (4) necessarily follow. We may state, therefore, the following theorem.

THEOREM 1. *A necessary and sufficient condition that a conformal transformation be a homographic transformation is that the associated function* $\lambda(x, y)$ *satisfy both of the following identities*

$$\lambda_{xx} - \lambda_{yy} \equiv 0, \qquad \lambda_{xy} \equiv 0.$$

A geometric interpretation of this condition is that at a general point w *of the curve* $\overline{C}$ *the rate of variation* $d\bar{\gamma}/d\bar{s}$ *of the curvature of* $\overline{C}$ *per unit length of arc* $\bar{s}$ *is independent of the direction of the curve* C *at the corresponding point* z.

Equation (3) shows that this is equivalent to the following characterization.

THEOREM 2. *The group of homographic transformations consists of all of the conformal transformations under which the differential form* $d\gamma ds$ *is absolutely invariant.*

The following theorem may be deduced, similarly, in consideration of equation (2).

THEOREM 3. *A necessary and sufficient condition that a transformation* (1) *be a directly conformal collineation is that the associated function* $\lambda(x, y)$ *satisfy both of the identities*

$$\lambda_x \equiv 0, \qquad \lambda_y \equiv 0.$$

Under this condition the differential form γds *is an absolute invariant of the transformation* (1).

Since $\gamma = d\theta/ds$ where $dz/ds = e^{i\theta}$, the differential form $d\gamma ds$ may be written in the form

$$(6)\qquad (d^2\theta/ds^2)(ds)^2$$

wherein $\theta = -i \log (dz/ds)$.

4. **The Schwarzian derivative.** Consider a curve defined by

$$z = x(t) + iy(t),$$

wherein x and y are functions of a real variable t. It is known that the Schwarzian derivative

$$\{z, t\} \equiv (d^3z/dt^3)/(dz/dt) - \tfrac{3}{2}[(d^2z/dt^2)/(dz/dt)]^2$$

is an absolute invariant under the homographic transformations. Let us investigate the geometric significance of the invariance of this derivative.

We find that

$$(7)\qquad z''/z' = i\gamma$$

where accents indicate differentiation with respect to s. On differentiating the members of equation (7) with respect to s we obtain

$$(8)\qquad z'''/z' - (z''/z')^2 \equiv i d\gamma/ds.$$

Making use of (7) and (8) we deduce

$$(9)\qquad \{z, s\} \equiv i d\gamma/ds + \gamma^2/2.$$

If we make a change of variable by the formula[2]

$$(10)\qquad \{z, t\} \equiv \{z, s\}(ds/dt)^2 + \{s, t\},$$

we obtain

$$(11)\qquad \{z, t\} \equiv (i d\gamma/ds + \gamma^2/2)(ds/dt)^2 + \{s, t\}.$$

The real and imaginary components of $\{z, t\}$,

$$R \equiv (\gamma^2/2)\ (ds/dt)^2 + \{s, t\},\qquad I \equiv (d\gamma/ds)\ (ds/dt)^2,$$

are, themselves, absolute invariants of the group of homographic transformations. Thus, if (1) is homographic, we have

$$(12)\qquad (\gamma^2/2)(ds/dt)^2 + \{s, t\} \equiv (\bar{\gamma}^2/2)(d\bar{s}/dt)^2 + \{\bar{s}, t\},$$

$$(13)\qquad (d\gamma/ds)(ds/dt)^2 \equiv (d\bar{\gamma}/d\bar{s})(d\bar{s}/dt)^2.$$

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[2] For the change of variable formula see, for example, Ford, *Automorphic Functions*, McGraw-Hill, 1929, p. 99.

If now we put $t=s$ in (12), we find

$$2\{\bar{s}, s\} \equiv \gamma^2 - \bar{\gamma}^2(d\bar{s}/ds)^2. \tag{14}$$

Likewise equation (12) yields

$$2\{s, \bar{s}\} \equiv \bar{\gamma}^2 - \gamma^2(ds/d\bar{s})^2 \tag{15}$$

on substituting $\bar{s}$ for t.

Equation (13) expresses the invariance of the form $d\gamma ds$. Making use of this equation and equations (14) and (15), we deduce

$$2\{\bar{s}, s\} \equiv \gamma^2 - \bar{\gamma}^2(d\gamma/d\bar{\gamma})^2, \tag{16}$$

$$2\{s, \bar{s}\} \equiv \bar{\gamma}^2 - \gamma^2(d\bar{\gamma}/d\gamma)^2. \tag{17}$$

These equations express the significance of the invariance of the Schwarzian derivatives $\{z, s\}$ and $\{z, \bar{s}\}$ as intrinsic geometric relations between any pair of curves C, $\bar{C}$ which correspond under a homographic transformation. To complete the geometric interpretations of (16) and (17) let us recall the significance of the Schwarzian derivative of a real function.[3] Consider two real functions $\sigma=\sigma(s)$ and $\bar{\sigma}=\bar{\sigma}(s)$ which are chosen to satisfy

$$\{\sigma, s\} \equiv \{\bar{\sigma}, s\}$$

identically in s. This relation is necessary and sufficient that $\sigma(s)$ and $\bar{\sigma}(s)$ be connected by a homographic transformation

$$\bar{\sigma}(s) = [a\sigma(s) + b]/[c\sigma(s) + d], \tag{18}$$

wherein a, b, c, d are constants and $ad-bc\neq 0$. The relation (18) is also necessary and sufficient that corresponding to any set of four values $s=s_j$, $(j=1, 2, 3, 4)$, the cross-ratios

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \qquad (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, \bar{\sigma}_4)$$

are identical. If as s varies, points P and $\bar{P}$ describe curves whose corresponding lengths of arc are defined by $\sigma=\sigma(s)$ and $\bar{\sigma}=\bar{\sigma}(s)$, the movements of these points will be called *projectively applicable*. This designation is suggested by the property that the development of these movements along a straight line produces projectively equivalent rectilinear movements.

Corresponding to a real single-valued differentiable function $\sigma=\sigma(s)$ there exists a *class* $\mathfrak{S}_{\sigma(s)}$ *of projectively applicable movements* to which a movement defined by $\bar{\sigma}=\bar{\sigma}(s)$ will be said to belong if $\{\bar{\sigma}, s\}\equiv\{\sigma, s\}$. We shall call the Schwarzian $\{\sigma, s\}$ *the absolute projective acceleration*

[3] Cartan, *Leçons sur la Théorie des Espaces*, Paris, Gauthier-Villars, 1937, p. 3.

of the movements of the class $\mathfrak{S}_{\sigma(s)}$ or simply the *absolute projective acceleration*[4] $\{\sigma, s\}$.

We may now state the following theorem:

THEOREM 4. *The invariance of the Schwarzian derivative* $\{z, t\}$ *yields the identities* (16) *and* (17) *which express the absolute projective accelerations* $\{\bar{s}, s\}$ *and* $\{s, \bar{s}\}$ *algebraically in terms of the squares of the curvatures* γ *and* $\bar{\gamma}$ *and the square of their rate of variation* $d\gamma/d\bar{\gamma}$.

Other interesting identities may be obtained by forming various combinations of (13), (14) and (15). One of these has the surprisingly simple form

$$\bar{\gamma}^2/\{s, \bar{s}\} + \gamma^2/\{\bar{s}, s\} \equiv 2. \tag{19}$$

5. **The magnimetric circles.** Let us consider a homographic transformation

$$w = (az + b)/(cz + d), \tag{20}$$

where a, b, c, d are constants and $ad-bc=1$. Since $\lambda(x, y)=|dz/dw|$ and for (20) dz/dw is defined by $dz/dw=(cz+d)^2$, we may write

$$\lambda(x, y) \equiv (cz + d)(\bar{c}\bar{z} + \bar{d}), \tag{21}$$

where $\bar{z}$, $\bar{c}$, $\bar{d}$ denote the conjugate imaginaries of z, c, d. Let the value of λ defined at the point $z_1=x_1+iy_1$ be denoted by λ_1. By making use of (20), (21) and the equation

$$z = (-dw + b)/(cw - a), \tag{22}$$

for the inverse of (20), the proof of the following theorem may be sup plied by the reader.

THEOREM 5. *Through a point* z_1, *in the z-plane (excluding* $z_1=\infty$ *and* $z_1=-d/c$), *there passes just one circle which by* (20) *is magnified in all of its elements of arc length by the constant multiple* $1/\lambda_1$. *Similarly, through the point* w_1, *which corresponds by* (20) *to the point* z_1, *there passes just one circle which by* (22) *is magnified in all of its elements of arc-length by the constant multiple* λ_1.

These circles will be called *magnimetric circles of the transformations* (20) *and* (22). The totality of the magnimetric circles in the z-plane form a family of concentric circles, a general one of which is defined by

$$(cz + d)(\bar{c}\bar{z} + \bar{d}) = \lambda, \qquad \lambda = \text{const.} \tag{23}$$

[4] Loc. cit., Footnote 3, p. 3. Cartan, in considering rectilinear motion defined by $x=x(t)$ has called the Schwarzian $\{x, t\}$ *"l'accélération projective du mouvement."*

The corresponding magnimetric circles in the w-plane form the family of concentric circles, a general one of which is defined by

$$(cw - a)(\bar{c}\bar{w} - \bar{a}) = 1/\lambda. \tag{24}$$

Let $r(z)$ and $\rho(w)$ denote the radii of the circles (23) and (24), respectively. We have, clearly, that

$$r^2(z) = \lambda/c\bar{c}, \qquad \rho^2(w) = 1/\lambda c\bar{c}. \tag{25}$$

When $\lambda = 1$, equations (24) and (25) represent the isometric[5] circles.

We shall refer to families in the z- and w-planes as z- and w-families, respectively. By making use of equations (23) and (24) in connection with (20), the following theorem is obtained.

THEOREM 6. *There are* ∞^2 *transformations of the form* (20) *which transform a z-family of concentric circles with an arbitrarily selected center* z_0 *into a w-family of concentric circles with an arbitrarily selected center* w_0. *There are* ∞^1 *of these which transform a selected circle of the z-family into a selected circle of the w-family. On requiring this pair of circles to correspond, a one-to-one correspondence among the other members of the two families is established. Finally, there is just one transformation of this infinite system which transforms a selected point* z_1 *on any circle of the z-family into a selected point* w_1 *on the corresponding circle of the w-family. The circles of these z- and w-families are the magnimetric circles of the transformation, and the product of any corresponding pair of radii is equal to* $1/|c|^2$.

UNIVERSITY OF KANSAS

[5] Loc. cit., Footnote 2, pp. 23–27.