CESÀRO SUMMABILITY OF A CLASS OF SERIES

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The following theorem has recently been proved by H. L. Garabedian:[1] If a_n is a polynomial in n of degree $k-1$, the series $\sum_{n=0}^{\infty}(-1)^n a_n$ is summable (C, k) but not $(C, k-1)$ to the value $\sum_{i=0}^{k-1}2^{-i-1}\Delta^i a_0$. Here and elsewhere in this paper k is understood to be a fixed positive integer.

Our present object is to obtain some extensions of this result. One of these may be stated at once, the proof being given at the end of the paper.

THEOREM 2. *Let a_n be a polynomial in n of degree $k-1$, and let z be a complex number such that $|z|=1$, $z\neq 1$. Then the series $\sum_{n=0}^{\infty}a_n z^n$ is summable (C, k) but not $(C, k-1)$ to the value $-\sum_{m=0}^{k-1}z^m(z-1)^{-m-1}\Delta^m a_0$.*

Before stating our first theorem we require the following definitions. Let f_n be a periodic function of the integer n; that is to say, let there be an integer p such that $f_{n+p}=f_n$ for all values of n. Let $M(f)$ denote the mean value of f_n over a period, thus $M(f)=(1/p)\sum_{n=0}^{p-1}f_n$. Suppose now that f_n is a periodic function of mean value zero; that is, let $M(f)=0$. Writing $f(n)$, $a(n)$ in place of f_n, a_n, set

$$
\text{(a)}\qquad
\begin{aligned}
f_1(n) &= \sum_{j=0}^{n} f(j), & c_1 &= M(f_1), & F_1(n) &= f_1(n) - c_1,\\
f_2(n) &= \sum_{j=0}^{n} F_1(j), & c_2 &= M(f_2), & F_2(n) &= f_2(n) - c_2,\\
f_3(n) &= \sum_{j=0}^{n} F_2(j), & c_3 &= M(f_3), & F_3(n) &= f_3(n) - c_3,
\end{aligned}
$$

and so on; this procedure ensures that $F_i(n)$ is periodic with mean value zero ($i=1, 2, \cdots$). In terms of these definitions we prove the following theorem:

THEOREM 1. *Let $a(n)$ be a polynomial in n of degree $k-1$, and $f(n)$ a periodic function with mean value zero. Then the series $\sum_{n=0}^{\infty}f(n)a(n)$ is summable (C, k) but not $(C, k-1)$ to the value $\sum_{i=0}^{k-1}c_{i+1}\Delta^i a_0$.*

If $f(n)=(-1)^n$ it is easily verified that $c_{i+1}=2^{-i-1}$, which shows

[1] This Bulletin, vol. 45 (1939), pp. 592–596. In the theorem as here stated it is required that a_n satisfy the conditions $\Delta^{k-1}a_0\neq 0$, $\Delta^i a_0=0$ $(i\geq k)$. It is easily seen that this is equivalent to requiring a_n to be a polynomial in n of degree $k-1$.

that the present theorem includes that of Garabedian as a special case. The method of proof is entirely different.

We shall make use of the notation of finite differences. We recall that $\Delta a(n)=a(n)-a(n+1)$, $\Delta^2 a(n)=\Delta(\Delta a(n))$, $\cdots$, and $E^j a(n)=a(n+j)$. We shall require the well known equation, in symbolic form,

$$\Delta^k = (1-E)^k = 1 - C_{k,1}E + C_{k,2}E^2 - \cdots + (-1)^k E^k, \tag{1}$$

where the $C_{k,i}$ are the binomial coefficients. We also write $n^{(k)}=n(n-1)(n-2)\cdots(n-k+1)$, and by convention $n^{(0)}=1$, $\Delta^0 a(n)=a(n)$; then the successive differences of $n^{(k)}$ are given by

$$\Delta^i n^{(k)} = (-1)^i k^{(i)} n^{(k-i)}, \qquad 0 \leqq i \leqq k. \tag{2}$$

We require two algebraic identities, which are stated as lemmas.

LEMMA 1. *We have, for r a positive integer,*

$$\sum_{n=0}^{r} f(n)a(n) = \sum_{i=0}^{k-1} c_{i+1}\Delta^i a(0) + \sum_{i=0}^{k-1} F_{i+1}(r)\Delta^i a(r+1) + \sum_{n=0}^{r} F_k(n)\Delta^k a(n).$$

For,

$$\begin{aligned}
\sum_{n=0}^{r} f(n)a(n) &= f(0)a(0) + f(1)a(1) + \cdots + f(r)a(r)\\
&= f_1(0)a(0) + [f_1(1) - f_1(0)]a(1) + \cdots\\
&\quad + [f_1(r) - f_1(r-1)]a(r)\\
&= [F_1(0) + c_1]a(0) + [F_1(1) - F_1(0)a](1) + \cdots\\
&\quad + [F_1(r) - F_1(r-1)]a(r)\\
&= c_1 a(0) + F_1(0)[a(0) - a(1)] + \cdots\\
&\quad + F_1(r-1)[a(r-1) - a(r)]\\
&\quad + F_1(r)[a(r) - a(r+1)] + F_1(r)a(r+1)\\
&= c_1 a(0) + F_1(r)a(r+1) + \sum_{n=0}^{r} F_1(n)\Delta a(n).
\end{aligned}$$

The lemma follows by repeated applications of this device.

LEMMA 2. *The expressions*

$$\frac{(r+1-n)(r+2-n)\cdots(r+k-n)}{(r+1)(r+2)\cdots(r+k)} \tag{3}$$

and

$$
(4) \qquad \begin{aligned}
1 &- C_{k,1}\frac{n}{r+1} + C_{k,2}\frac{n(n+1)}{(r+1)(r+2)} \\
&- C_{k,3}\frac{n(n+1)(n+2)}{(r+1)(r+2)(r+3)} + \cdots \\
&+ (-1)^k \frac{n(n+1)\cdots(n+k-1)}{(r+1)(r+2)\cdots(r+k)}
\end{aligned}
$$

are algebraically identical.

To prove this, set

$$
G(t) = \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t)}, \qquad t \text{ a positive integer,}
$$

$$
G(0) = 1.
$$

Then, if $EG(t)=G(t+1)$, the expression (4) may be written

$$
\begin{aligned}
G(0) - C_{k,1}EG(0) + C_{k,2}E^2G(0) - \cdots + (-1)^kE^kG(0) \\
= (1-E)^kG(0) = \Delta^kG(0),
\end{aligned}
$$

by (1). We have therefore to show that $\Delta^kG(0)$ is equal to the expression (3). We write

$$
\begin{aligned}
\Delta G(t) &= \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t)} - \frac{n(n+1)\cdots(n+t)}{(r+1)(r+2)\cdots(r+t+1)} \\
&= \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t+1)}(r+1-n), \qquad t \geqq 1.
\end{aligned}
$$

In particular

$$
\Delta G(1) = \frac{n}{(r+1)(r+2)}(r+1-n).
$$

Also, since $G(0)=1$ and since $G(1)=n/(r+1)$, we have $\Delta G(0) =(r+1-n)/(r+1)$. Similarly we find

$$
\Delta^2G(t) = \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t+2)}(r+1-n)(r+2-n),
$$

$$
\Delta^2G(1) = \frac{n}{(r+1)(r+2)(r+3)}, \quad \Delta^2G(0) = \frac{(r+1-n)(r+2-n)}{(r+1)(r+2)}.
$$

Proceeding in this way, we obtain the desired result by induction.

PROOF OF THEOREM 1. The series $\sum_{n=0}^{\infty} f(n)a(n)$ is said to converge (C, k) to the value S provided the expression

$$\sum_{n=0}^{r} \frac{(r+1-n)(r+2-n)\cdots(r+k-n)}{(r+1)(r+2)\cdots(r+k)} f(n)a(n)$$

tends to S as r becomes infinite. Now any polynomial $a(n)$ of degree $k-1$ can be expressed in the form $A_1(n-1)^{(k-1)}+A_2(n-1)^{(k-2)}+\cdots$ where $A_1, A_2, \cdots$ are constants. Hence we need only consider

$$\sum_{n=0}^{r} \frac{(r+1-n)(r+2-n)\cdots(r+k-n)}{(r+1)(r+2)\cdots(r+k)} (n-1)^{(k-1)} f(n),$$

which by Lemma 2 is equal to

$$\sum_{n=0}^{r}\left\{1-C_{k,1}\frac{n}{r+1}+C_{k,2}\frac{n(n+1)}{(r+1)(r+2)}-\cdots\right\}(n-1)^{(k-1)}f(n). \tag{5}$$

We consider the terms of this expression separately. By (2) and Lemma 1 we have

$$\begin{aligned}
\sum_{n=0}^{r}(n-1)^{(k-1)}f(n) &= \sum_{i=0}^{k-1} c_{i+1}(-1)^i(k-1)^{(i)}(-1)^{(k-1-i)} \\
&\quad + \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i(k-1)^{(i)} r^{(k-1-i)}, \\
\sum_{n=0}^{r} n^{(k)}f(n) &= \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i k^{(i)}(r+1)^{(k-i)} \\
&\quad + (-1)^k k! \sum_{n=0}^{r} F_k(n), \\
\sum_{n=0}^{r}(n+1)^{(k+1)}f(n) &= \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i(k+1)^{(i)}(r+2)^{(k+1-i)} \\
&\quad + (-1)^k(k+1)^{(k)}\sum_{n=0}^{r}(n+1)F_k(n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots, \\
\sum_{n=0}^{r}(n+k-1)^{(2k-1)}f(n) &= \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i(2k-1)^{(i)}(r+k)^{(2k-1-i)} \\
&\quad + (-1)^k(2k-1)^{(k)}\sum_{n=0}^{r}(n+k-1)^{(k-1)}F_k(n).
\end{aligned}$$

Multiply the above equations by

$$1, \quad -\frac{C_{k,1}}{r+1}, \quad \frac{C_{k,2}}{(r+1)(r+2)}, \cdots,$$

respectively, and add. We then have on the left the expression (5) and on the right

$$\sum_{i=0}^{k-1} c_{i+1}(-1)^i(k-1)^{(i)}(-1)^{(k-1-i)} \\ + \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i r^{(k-1-i)}(1-E)^k(k-1)^{(i)} \\ + \text{terms involving } \sum_{n=0}^{r}.$$

Now, by (1), $(1-E)^k(k-1)^{(i)} = \Delta^k(k-1)^{(i)} = 0$ for $0 \leqq i \leqq k-1$. Also, the terms involving $\sum_{n=0}^{r}$ can easily be shown to approach zero as r becomes infinite. We have therefore proved that the series $\sum_{n=0}^{\infty}(n-1)^{(k-1)}f(n)$ converges (C, k) to the value $\sum_{i=0}^{k-1} c_{i+1}(-1)^i \cdot (k-1)^{(i)}(-1)^{(k-1-i)}$, or, setting $b(n) = (n-1)^{(k-1)}$, $\sum_{n=0}^{\infty} b(n)f(n)$ converges (C, k) to the value $\sum_{i=0}^{k-1} c_{i+1}\Delta^i b(0)$.

Theorem 1 now follows without difficulty. It is easily seen on examining the argument that no order of summation less than k will serve.

We now prove Theorem 2. If am z is a rational multiple of π, say am $z = 2\pi q/p$, where p and q are integers, the function $f(n) = z^n$ is periodic with period p and mean value zero. It is easily verified that, for $m \geqq 0$, the equations (a) become

$$\text{(b)} \qquad f_{m+1}(n) = \frac{z^m}{(z-1)^m}\,\frac{z^{n+1}-1}{z-1}, \qquad c_{m+1} = -\frac{z^m}{(z-1)^{m+1}}, \\ F_{m+1}(n) = \frac{z^{m+1}}{(z-1)^{m+1}}\, z^n,$$

whence Theorem 2 follows at once, as a special case of Theorem 1. If am z is not a rational multiple of π, then we define $M(f)$ as $\lim_{r\to\infty}(1/r)\sum_{n=0}^{r=1} f(n)$. Then the equations (b) still hold, and the theorem follows by the same argument as Theorem 1.

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