A SEQUENCE OF LIMIT TESTS FOR THE CONVERGENCE OF SERIES[1]

MARGARET MARTIN

In this paper, we shall develop a sequence of limit tests for the convergence and divergence of infinite series of positive terms which is similar in form to the De Morgan and Bertrand sequence but involves the ratio of two successive values of the test ratio rather than the test ratio itself. The proof will be based on the following integral test by R. W. Brink:[2]

"THEOREM VI. *Given the sequence* $\{u_n\}$. *Let* $r_n = u_{n+1}/u_n$ *and* $R_n = r_{n+1}/r_n = u_{n+2}u_n/u_{n+1}^2$. *If* $\lim_{n=\infty} r_n = 1$, *and if* $R(x)$ *is a function such that* $R(n) = R_n$, *and such that* $R(x) \geqq R(x')$ *when* $x' > x$, *a necessary and sufficient condition for the convergence of the series* $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral*

$$\int_0^{\infty} \exp\left\{-\int_0^x \int_x^{\infty} \log R(x)\, dx dx\right\} dx."$$

Since a finite number of terms does not affect convergence or divergence, the conditions of Theorem VI need hold only for n greater than some fixed number ν, in which case zero is to be replaced by ν as a lower limit of integration.

The foregoing theorem admits a generalization similar to that given by C. T. Rajagopal[3] in the case of another theorem of Brink's.[4] However, Brink's Theorem VI is sufficient for the purposes of the present paper.

LEMMA. *Let* $\{u_n\}$ *and* $\{u_n'\}$ *be sequences of positive terms with ratios* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, $r_n' = u_{n+1}'/u_n'$, *and* $R_n' = r_{n+1}'/r_n'$, *such that* $\lim_{n\to\infty} r_n = \lim_{n\to\infty} r_n' = 1$.

1. *If the series* $\sum_{n=\nu}^{\infty} u_n'$ *converges and if* $R_n \geqq R_n'$ *for all values of* $n \geqq \nu$, *then the series* $\sum_{n=\nu}^{\infty} u_n$ *converges.*

2. *If the series* $\sum_{n=\nu}^{\infty} u_n'$ *diverges and if* $R_n \leqq R_n'$ *for all values of* $n \geqq \nu$, *then the series* $\sum_{n=\nu}^{\infty} u_n$ *diverges.*

[1] Presented to the Society, April 27, 1940, under the title *A sequence of tests for the convergence and divergence of infinite series.*

[2] R. W. Brink, *A new sequence of integral tests for the convergence and divergence of infinite series,* Annals of Mathematics, vol. 21 (1919), pp. 39–60.

[3] C. T. Rajagopal, *On an integral test of R. W. Brink for the convergence of series,* this Bulletin, vol. 43 (1937), pp. 405–412.

[4] R. W. Brink, *A new integral test for the convergence and divergence of infinite series,* Transactions of this Society, vol. 19 (1918), pp. 186–204.

PROOF. In case 1, if $n \geqq \nu$,

$$\frac{r_{n+1}}{r_n} \geqq \frac{r'_{n+1}}{r'_n}, \quad \frac{r_{n+2}}{r_{n+1}} \geqq \frac{r'_{n+2}}{r'_{n+1}}, \cdots, \quad \frac{r_{N+1}}{r_N} \geqq \frac{r'_{N+1}}{r'_N}.$$

Multiplying these inequalities, we have $r_{N+1}/r_n \geqq r'_{N+1}/r'_n$, and taking the limit as N becomes infinite, we obtain $r_n \leqq r'_n$, $(n \geqq \nu)$. Hence if $\sum_{n=\nu}^{\infty} u'_n$ converges, $\sum_{n=\nu}^{\infty} u_n$ also converges. A similar proof can be given for the case of divergence.

THEOREM 1. *Let* $\{u_n\}$ *be a sequence of positive terms with ratios* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, *such that* $\lim_{n\to\infty} r_n = 1$. *If of the limits*

$$\lim_{n\to\infty} n^2 \log R_n = a_0,$$

$$\lim_{n\to\infty} \log n(n^2 \log R_n - 1) = a_1,$$

$$\lim_{n\to\infty} \log \log n\{\log n(n^2 \log R_n - 1) - 1\} = a_2,$$

$$\lim_{n\to\infty} \log \log \log n[\log \log n\{\log n(n^2 \log R_n - 1) - 1\} - 1] = a_3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

a_k *is the first which is finite and different from* 1, *or the first to be positively or negatively infinite, the series* $\sum_{n=\nu}^{\infty} u_n$ *converges if* $a_k > 1$ *and diverges if* $a_k < 1$.

PROOF. Let $l_1 = \log n$, $l_k = \log l_{k-1}$, $(k > 1)$;

$$L_k(n, \alpha) = \frac{1}{l_1} + \frac{1}{l_1 l_2} + \cdots + \frac{1}{l_1 l_2 \cdots l_{k-1}} + \frac{\alpha}{l_1 l_2 \cdots l_k}, \quad k \geqq 1,$$

$$L_0(n, \alpha) = 0.$$

By hypothesis,

$$(1) \quad \lim_{n\to\infty} l_k[l_{k-1}\{l_{k-2}(\cdots l_2[l_1(n^2 \log R_n - 1) - 1] \cdots) - 1\} - 1] = a_k.$$

Hence

$$(2) \quad \begin{aligned} &\lim_{n\to\infty} \frac{1}{1+L_k(n, 1)} [l_k\{l_{k-1}(l_{k-2}[\cdots l_2\{l_1(n^2 \log R_n - 1) - 1\} \cdots] - 1) - 1\} \\ &\quad - l_k l_{k-1} \cdots l_2 L_1(n, 1) - l_k l_{k-1} \cdots l_3 L_2(n, 1) - \cdots - l_k l_{k-1} L_{k-2}(n, 1) \\ &\quad - l_k L_{k-1}(n, 1)] = a_k, \end{aligned}$$

since $\lim_{n\to\infty} 1/[1 + L_k(n, 1)] = 1$ and $\lim_{n\to\infty} l_k l_{k-1} \cdots l_j L_{j-1}(n, 1) = 0$, $(1 \leqq j \leqq k)$.

(a) If $a_k>1$, let α_1 be a number such that $1<\alpha_1<a_k$. Let N_1 be chosen sufficiently large so that for $n\geqq N_1$, $l_k(n)$ is defined and positive and

$$\begin{aligned}(3)\quad &\frac{1}{1+L_k(n,1)}[l_k\{l_{k-1}(l_{k-2}[\cdots l_2\{l_1(n^2\log R_n-1)-1\}\cdots]-1)-1\}\\ &-l_kl_{k-1}\cdots l_2L_1(n,1)-l_kl_{k-1}\cdots l_3L_2(n,1)-\cdots-l_kl_{k-1}L_{k-2}(n,1)\\ &-l_kL_{k-1}(n,1)]>\alpha_1.\end{aligned}$$

It follows that

$$\begin{aligned}\log R_n &> \frac{1}{n^2}+\frac{1}{n^2l_1}[1+L_1(n,1)]+\frac{1}{n^2l_1l_2}[1+L_2(n,1)]+\cdots\\ (4)\quad &+\frac{1}{n^2l_1l_2\cdots l_{k-1}}[1+L_{k-1}(n,1)]\\ &+\frac{\alpha_1}{n^2l_1l_2\cdots l_k}[1+L_k(n,1)].\end{aligned}$$

Let

$$\begin{aligned}M_k(x,\alpha) &= \frac{1}{x^2}+\frac{1+l_1}{x^2l_1^2}+\frac{1+l_2+l_2l_1}{x^2l_1^2l_2^2}+\cdots\\ &+\frac{1+l_{k-1}+l_{k-1}l_{k-2}+\cdots+l_{k-1}\cdots l_1}{x^2l_1^2l_2^2\cdots l_{k-1}^2}\\ &+\alpha\frac{1+l_k+l_kl_{k-1}+\cdots+l_k\cdots l_1}{x^2l_1^2l_2^2\cdots l_k^2},\end{aligned}$$

where now $l_1=\log x$, and so on. Then (4) can be written

$$(5)\qquad \log R_n > M_k(n,\alpha_1).$$

(b) If $a_k<1$, let α_2 be a positive number such that $a_k<\alpha_2\leqq 1$. Proceeding as in (a), we can show that for n greater than a suitably chosen number N_2,

$$(6)\qquad \log R_n < M_k(n,\alpha_2).$$

Consider the series

$$(7)\qquad \sum_{n=\nu}^{\infty}u'_{n;\alpha}\qquad u'_{n;\alpha}=\exp\left\{-\sum_{j=\nu}^{n-1}\sum_{i=j}^{\infty}M_k(i,\alpha)\right\},\qquad \nu>N_1,N_2.$$

For this series, $r'_n=\exp\{-\sum_{i=n}^{\infty}M_k(i,\alpha)\}$, $R'_n=\exp\{M_k(n,\alpha)\}$. It is easily shown that the conditions of Brink's Theorem VI are satisfied with $R'(x)=\exp\{M_k(x,\alpha)\}$ and $n\geqq\nu$. The test integral then has the form

$$\int_\nu^\infty \exp\left\{-\int_\nu^x\int_x^\infty M_k(x,\alpha)dxdx\right\}dx = K\int_\nu^\infty \frac{1}{xl_1l_2\cdots l_{k-1}l_k^\alpha}dx,$$

where K is constant. This integral, and hence the series $\sum_{n=\nu}^\infty u'_{n;\alpha}$, converges for $\alpha>1$ and diverges for $\alpha\leqq 1$. We now apply the lemma to the series $\sum_{n=\nu}^\infty u_n$ and $\sum_{n=\nu}^\infty u'_{n;\alpha}$.

In case (a), we set $\alpha=\alpha_1$ in series (7). $\sum_{n=\nu}^\infty u'_{n;\alpha_1}$ converges since $\alpha_1>1$. From (5), we have $R_n>\exp\{M_k(n,\alpha_1)\}=R_n$, $n\geqq\nu$. The conditions of part 1 of the lemma are satisfied, and hence $\sum_{n=\nu}^\infty u_n$ converges.

In case (b), we set $\alpha=\alpha_2$ in series (7). $\sum_{n=\nu}^\infty u'_{n;\alpha_2}$ diverges, and from (6),

$$R_n < \exp\{M_k(n,\alpha_2)\} = R'_n, \qquad n\geqq\nu.$$

Hence $\sum_{n=\nu}^\infty u_n$ diverges by part 2 of the lemma.

The tests of Theorem 1 apply to series for which an explicit expression for R_n is known. The general term of such a series has the form $u_n=\prod_{j=\nu}^{n-1}\prod_{m=j}^\infty\phi(m)$.

Example 1. Consider the series $\sum_{n=3}^\infty u_n$, where

$$u_n = \exp\left\{-\sum_{j=2}^{n-1}\sum_{m=j}^\infty\phi(m)\right\}, \qquad \phi(m) = \frac{\alpha+\beta\log(m^2)}{m^2\log(m^2)}, \qquad \beta>0.$$

We have $r_n=\exp\{-\sum_{m=n}^\infty\phi(m)\}$; $\lim_{n\to\infty}r_n=1$ since $\sum_{m=3}^\infty\phi(m)$ converges for all values of α and β; $R_n=\exp\{\phi(n)\}$; $\log R_n=\phi(n)$. We apply the first test of Theorem 1,

$$\lim_{n\to\infty} n^2\log R_n = \lim_{n\to\infty}\frac{\alpha+\beta\log(n^2)}{\log(n^2)} = \beta.$$

Thus the series converges for $\beta>1$ and diverges for $\beta<1$, regardless of the value of α. For the case $\beta=1$, we apply the second test,

$$\lim_{n\to\infty}\log n(n^2\log R_n - 1) = \lim_{n\to\infty}\log n\left[\frac{\alpha+\log(n^2)}{\log(n^2)}-1\right] = \frac{\alpha}{2},$$

and the series converges for $\beta=1$, $\alpha>2$, and diverges for $\beta=1$, $\alpha<2$. For the case $\beta=1$, $\alpha=2$, we go on to the third limit test,

$$\lim_{n\to\infty}\log\log n[\log n(n^2\log R_n-1)-1] = 0,$$

and the series diverges.

The tests of Theorem 1 are valid if $\log R_n$ is replaced by $R_n - 1$, the tests of the resulting sequence being in some cases more convenient to apply than those of Theorem 1.

THEOREM 2. *Let* $\{u_n\}$ *be a sequence of positive terms with ratios* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, *such that* $\lim_{n\to\infty} r_n = 1$. *If of the limits*

$$\lim_{n\to\infty} n^2(R_n - 1) = b_0,$$

$$\lim_{n\to\infty} \log n[n^2(R_n - 1) - 1] = b_1,$$

$$\lim_{n\to\infty} \log\log n\{\log n[n^2(R_n - 1) - 1] - 1\} = b_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

b_k *is the first which is finite and different from* 1, *or the first to be positively or negatively infinite, the series* $\sum_{n=\nu}^{\infty} u_n$ *converges if* $b_k > 1$ *and diverges if* $b_k < 1$.

PROOF. Proceeding as in the proof of Theorem 1, we are led to the following inequalities:

(a) If $b_k > 1$, then for any number β_1 such that $1 < \beta_1 < b_k$, and for n greater than a suitably chosen number N_1',

$$R_n - 1 > M_k(n, \beta_1). \tag{8}$$

(b) If $b_k < 1$, then for any positive number β_2 such that $b_k < \beta_2 \leqq 1$, and for n greater than a suitably chosen number N_2',

$$R_n - 1 < M_k(n, \beta_2). \tag{9}$$

In case (a), consider the series $\sum_{n=\nu}^{\infty} u_n''$,

$$u_n'' = \exp\left\{-\sum_{j=\nu}^{n-1}\sum_{i=j}^{\infty}[M_k(i, \beta_1) - (M_k(i, \beta_1))^2]\right\}, \qquad \nu \geqq N_1'.$$

It can be shown that this series satisfies the conditions of Brink's Theorem VI with $R''(x) = \exp\{M_k(x, \beta_1) - (M_k(x, \beta_1))^2\}$. The test integral has the form

$$\int_\nu^\infty \exp\left\{-\int_\nu^x\int_x^\infty [M_k(x, \beta_1) - (M_k(x, \beta_1))^2]dxdx\right\}dx$$

$$\leqq \int_\nu^\infty \exp\left\{-\int_\nu^x\int_x^\infty M_k(x, \beta_1)dxdx + \int_\nu^\infty\int_x^\infty (M_k(x, \beta_1))^2 dxdx\right\}dx$$

$$= K'\int_\nu^\infty \exp\left\{-\int_\nu^x\int_x^\infty M_k(x, \beta_1)dxdx\right\}dx,$$

where K' is a constant. Since $\beta_1 > 1$, the latter integral converges, and hence the series $\sum_{n=\nu}^{\infty} u_n''$ converges. From (8),

$$R_n > 1 + M_k(n, \beta_1), \qquad n \geqq \nu.$$

Hence if ν is sufficiently large so that $M_k(\nu, \beta_1) < 1$,

$$\log R_n > \log [1 + M_k(n, \beta_1)] > M_k(n, \beta_1) - (M_k(n, \beta_1))^2 = \log R_n'', \qquad n \geqq \nu,$$

$R_n > R_n''$, and the series $\sum_{n=\nu}^{\infty} u_n$ of our theorem converges by part 1 of the lemma.

In case (b), set $\alpha = \beta_2$ in series (7), and take $\nu = N_2'$ and sufficiently large so that $|R_n' - 1| < 1$ for $n \geqq \nu$. $\sum_{n=\nu}^{\infty} u'_{n;\beta_2}$ diverges since $\beta_2 \leqq 1$. From (9), we have

$$R_n - 1 < M_k(n, \beta_2) = \log R_n' < R_n' - 1, \qquad n \geqq \nu,$$

$R_n < R_n'$, and hence the series $\sum_{n=\nu}^{\infty} u_n$ diverges by part 2 of the lemma.

Example 2. Consider the series $\sum_{n=3}^{\infty} u_n$, $u_n = \prod_{k=2}^{n-1} \prod_{m=k}^{\infty} (1 - \alpha/m^\beta)$, $(\alpha > 0, \beta > 1)$. Here

$$r_n = \sum_{m=n}^{\infty} \left(1 - \frac{\alpha}{m^\beta}\right), \qquad R_n = \frac{1}{1 - \alpha/n^\beta}, \qquad R_n - 1 = \frac{\alpha}{n^\beta - \alpha}.$$

$\operatorname{Lim}_{n\to\infty} r_n = 1$, since $\prod_{m=3}^{\infty}(1 - \alpha/m^\beta)$ converges. Applying the first test of Theorem 2, we find

$$\lim_{n\to\infty} n^2(R_n - 1) = \lim_{n\to\infty} n^2 \cdot \frac{\alpha}{n^\beta - \alpha} = \begin{cases} +\infty, & \beta < 2, \\ 0, & \beta > 2, \\ \alpha, & \beta = 2. \end{cases}$$

Thus the series converges when $1 < \beta < 2$ and diverges when $\beta > 2$. If $\beta = 2$, the series converges for $\alpha > 1$ and diverges for $\alpha < 1$. If $\beta = 2$ and $\alpha = 1$, we apply the second test of the sequence,

$$\lim_{n\to\infty} \log n [n^2(R_n - 1) - 1] = \lim_{n\to\infty} \log n \left[n^2 \frac{1}{n^2 - 1} - 1 \right] = 0,$$

and hence the series diverges.

UNIVERSITY OF MINNESOTA