SOME RECENT DEVELOPMENTS IN THE THEORY OF CONTINUED FRACTIONS[1]

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The principal rôle of the continued fraction in analysis has perhaps been that of an intermediary between more familiar and easily handled things, such as between the power series and the integral. This may partly explain the fact that there are certain questions about continued fractions which have remained relatively unexplored. To illustrate what I mean, if one's principal attention were focused upon power series, and continued fractions were used only incidentally, it is unlikely that one would imagine that the convergence region for the continued fraction is, as it now appears, more properly a *parabolic* region than a circular region.

I wish to speak today about some results which have been obtained during the last few years, by a group of men with whom I have been associated. Our investigations have been centered mainly upon the continued fraction itself. In certain instances it has been possible to apply our results to problems not directly connected with continued fractions. Thus, during the course of this lecture I shall have occasion to speak of the problem of moments, of Hausdorff summability, and of certain classes of analytic functions.

1. **Some definitions and formulas.** Before discussing some of the problems with which we have been concerned, I shall put down some necessary definitions and formulas. The continued fractions considered are chiefly of the form

$$\frac{1}{1}+\frac{a_2}{1}+\frac{a_3}{1}+\frac{a_4}{1}\quad\cdots \tag{1.1}$$

in which $a_2, a_3, a_4, \cdots$ are complex numbers. Apart from unessential initial irregularities, any continued fraction in which the partial denominators are different from zero can be thrown into this form. The nth *approximant* of (1.1) is the ordinary fraction, A_n/B_n, obtained by stopping with the nth partial quotient. The *numerators* and *denominators* may be computed by means of the recursion formulas

$$\begin{aligned} &A_0 = 0, \quad B_0 = 1, \quad A_1 = 1, \quad B_1 = 1, \\ &A_n = A_{n-1} + a_nA_{n-2}, \quad B_n = B_{n-1} + a_nB_{n-2}, \quad n = 2, 3, 4, \cdots . \end{aligned} \tag{1.2}$$

[1] An address delivered before the Detroit meeting of the Society, November 23, 1940, by invitation of the Program Committee.

An important consequence of the recursion formulas is the determinant formula

$$(1.3) \qquad A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}a_2a_3 \cdots a_n.$$

To assign a meaning to the continued fraction, we suppose, first, that $B_n \neq 0$ from and after some n. If, then, the limit $\lim_{n=\infty} (A_n/B_n)$ exists and is finite, the continued fraction is said to converge, and the value of this limit is the *value* of the continued fraction. The convergence problem is the problem of determining conditions upon $a_2, a_3, a_4, \cdots$ which are sufficient for the convergence of the continued fraction. For instance, one may determine a "convergence region" in which $a_2, a_3, a_4, \cdots$ may vary independently and the continued fraction remains convergent. Some idea of the nature of this problem may be had from the fact that it is possible to choose $a_2, a_3, a_4, \cdots$ in such a way that these numbers form an everywhere dense set in the complex plane, and the continued fraction is convergent.[2] Worpitzky[3] showed that (1.1) converges if $|a_n| \leqq 1/4$. This is the best circular convergence region with center at the origin, inasmuch as (1.1) diverges if $a_n = -\frac{1}{4} - c$, $c > 0$. Szász[4] found that if a is not a real number $\leqq -\frac{1}{4}$, then there exists a positive number r such that (1.1) converges if $|a_n - a| \leqq r$. We[5] found recently that if $R(a) > -1/4$ r may be taken equal to $\frac{1}{4}(|1+2a| - 2|a|)$. Again, if $|a_{2n+1}| \leqq 1/4$, $|a_{2n}| \geqq 25/4$, (1.1) is convergent.[6]

There is one important *necessary* condition for convergence due to von Koch,[7] which applies whenever the partial numerators are different from zero. Write the continued fraction in the form

$$(1.4) \qquad \frac{1}{1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} + \cdots$$

where $a_n = 1/b_{n-1}b_n$, $(n = 2, 3, 4, \cdots, b_1 = 1)$. Then, if the series $\sum |b_n|$ *converges*, the sequences of even and odd approximants have distinct

[2] Walter Leighton and H. S. Wall, *On the transformation and convergence of continued fractions* (to be referred to as T), American Journal of Mathematics, vol. 58 (1936), pp. 267–281, p. 269.

[3] Worpitzky, Jahresbericht, Friedrichs-Gymnasium und Realschule, Berlin, 1865, pp. 3–39. Independent proofs of this theorem were given later by Pringsheim and by Van Vleck.

[4] O. Perron, *Die Lehre von den Kettenbrüchen* (to be referred to as "Perron"), 2d edition, Leipzig and Berlin, 1929, p. 282.

[5] W. T. Scott and H. S. Wall, *A convergence theorem for continued fractions* (to be referred to as CT), Transactions of this Society, vol. 47 (1940), pp. 155–172, p. 171.

[6] T, p. 278.

[7] Perron, pp. 235–236.

limits (one may be infinite), and the continued fraction diverges by oscillation.

This natural breaking up of the sequence of approximants into two sequences makes it often of importance to consider the continued fraction

$$(1.5)\qquad \frac{1}{1 + a_2 -}\ \frac{a_2a_3}{1 + a_3 + a_4 -}\ \frac{a_4a_5}{1 + a_5 + a_6 -}\ \cdots$$

having A_{2n}/B_{2n} as its nth approximant; and the continued fraction

$$(1.6)\quad 1 - \frac{a_2}{1 + a_2 + a_3 -}\ \frac{a_3a_4}{1 + a_4 + a_5 -}\ \frac{a_5a_6}{1 + a_6 + a_7 -}\ \cdots,$$

having A_{2n+1}/B_{2n+1} as its nth approximant.[8] We shall refer to these as the *even part* and the *odd part*, respectively, of (1.1).

2. **Some ideas on convergence.** A natural procedure to follow in attacking the convergence problem is to turn to the infinite series

$$\frac{A_1}{B_1} + \left(\frac{A_2}{B_2} - \frac{A_1}{B_1}\right) + \left(\frac{A_3}{B_3} - \frac{A_2}{B_2}\right) + \cdots,$$

which, by (1.3), may be written in the form

$$(2.1)\qquad 1 - \frac{a_2}{B_1B_2} + \frac{a_2a_3}{B_2B_3} - \frac{a_2a_3a_4}{B_3B_4} + \cdots.$$

The starting point in our investigation is the idea of forming a majorant for the series (2.1) by requiring that the test-ratio

$$\rho_n = \frac{(-1)^n a_2a_3 \cdots a_{n+1}}{B_nB_{n+1}} : \frac{(-1)^{n-1}a_2a_3 \cdots a_n}{B_{n-1}B_n} = \frac{-a_{n+1}B_{n-1}}{B_{n+1}}$$

remain numerically less than or equal to some number r_n:

$$(2.2)\qquad |\rho_n| \leqq r_n, \qquad n = 1, 2, 3, \cdots.$$

If this holds, then it is necessarily true that $B_n \neq 0$, $(n = 1, 2, 3, \cdots)$. Hence the continued fraction converges if some a_n vanishes, or, in any case, if the majorant series $1 + \sum r_1r_2 \cdots r_n$ converges. Moreover, the sum of this majorant series is an upper bound for the absolute value of the continued fraction. If the a_n's are variables, and the r_n's are independent of the a_n's when the latter lie in a certain domain,

[8] Perron, p. 201.

then the continued fraction converges uniformly over this domain if the majorant series converges.

Now, in order to translate the condition (2.2) into a condition upon $a_2, a_3, a_4, \cdots$ we have the simple relation

$$\rho_n = \frac{-a_{n+1}(1+\rho_{n-1})}{1+a_{n+1}(1+\rho_{n-1})}, \qquad n = 1, 2, 3, \cdots, \rho_0 = 0. \tag{2.3}$$

Using this, the desired translations may be made in a great many different ways. We have found[9] the following to be especially useful:

$$r_n\left|1+a_n+a_{n+1}\right| \geqq r_n r_{n-2}\left|a_n\right| + \left|a_{n+1}\right|, \tag{2.4}$$

$(n=1, 2, 3, \cdots)$, where we agree to put $r_{-1}=r_0=a_1=0$. If, for example, we take $r_n=n/(n+2)$, so that the majorant series $1+\sum r_1r_2\cdots r_n=2$, we find at once the theorem of Worpitsky mentioned in §1, namely: (1.1) converges (uniformly) for $|a_n|\leqq 1/4$, $(n=2, 3, 4, \cdots)$. The upper bound 2 which we find in this case is the least upper bound inasmuch as the value of (1.1) is 2 when $a_n=-1/4$.

Again, by making an appropriate choice of the r_n's, one can derive the theorem that the continued fraction

$$\frac{g_1}{1+}\,\frac{(1-g_1)g_2x_2}{1+}\,\frac{(1-g_2)g_3x_3}{1+}\cdots \tag{2.5}$$

in which $0<g_1<1$, $0\leqq g_n<1$, $(n=2, 3, 4, \cdots)$, converges uniformly for $|x_n|\leqq 1$, $(n=2, 3, 4, \cdots)$, and its absolute value in this domain does not exceed

$$1 - \cfrac{1}{1+\displaystyle\sum_{n=1}^{\infty}\frac{g_1g_2\cdots g_n}{(1-g_1)(1-g_2)\cdots(1-g_n)}}. \tag{2.6}$$

This value is actually attained by the continued fraction if $x_n=-1$. We thus improve in important respects the well known *Pringsheim criteria*. An immediate consequence is the theorem that

$$\frac{1}{1+}\,\frac{g_1x_1}{1+}\,\frac{(1-g_1)g_2x_2}{1+}\,\frac{(1-g_2)g_3x_3}{1+}\cdots \tag{2.7}$$

converges uniformly for $|x_n|\leqq 1$, $(n=1, 2, 3, \cdots)$, if the series in (2.6) converges; and for $|x_1|\leqq h<1$, $|x_n|\leqq 1$, $(n=2, 3, 4, \cdots)$, if this same series diverges or converges. Part of this theorem was

[9] CT, p. 155 ff.

proved by E. B. Van Vleck,[10] who seems to have been the first to recognize the importance of continued fractions of this form.

The relationship of the inequalities (2.4) to continued fractions of the form (2.7) is indicated by the following theorem.

THEOREM 2.1. *If the inequalities* (2.4) *hold with actual inequality for* $n=1, 2$, *and if* $a_n\neq 0$, $(n=2, 3, 4, \cdots)$, *then the even and odd parts of* (1.1) *can, except for initial irregularities, be written in the form* (2.7). *In fact, the even part can be put into the form*

$$\frac{1}{1+a_2}\left\{\frac{1}{1}\underset{+}{}\frac{g_1x_1}{1}\underset{+}{}\frac{(1-g_1)g_2x_2}{1}\underset{+}{}\frac{(1-g_2)g_3x_3}{1}\underset{+}{}\cdots\right\} \tag{2.8}$$

where $|x_1|<1$, $|x_n|\leqq 1$, $(n=2, 3, 4, \cdots)$, $0<g_n<1$, $(n=1, 2, 3, \cdots)$; *and the odd part can be put into the form*

$$1-\frac{a_2}{1+a_2+a_3}\left\{\frac{1}{1}\underset{+}{}\frac{h_1y_1}{1}\underset{+}{}\frac{(1-h_1)h_2y_2}{1}\underset{+}{}\frac{(1-h_2)h_3y_3}{1}\underset{+}{}\cdots\right\}, \tag{2.9}$$

where $|y_1|<1$, $|y_n|\leqq 1$, $(n=2, 3, 4, \cdots)$, $0<h_n<1$, $(n=1, 2, 3, \cdots)$.

Put

$$g_n=\frac{r_{2n+1}|1+a_{2n+1}+a_{2n+2}|-|a_{2n+2}|}{r_{2n+1}|1+a_{2n+1}+a_{2n+2}|},$$

$$h_n=\frac{r_{2n+2}|1+a_{2n+2}+a_{2n+3}|-|a_{2n+3}|}{r_{2n+2}|1+a_{2n+2}+a_{2n+3}|}.$$

It follows from the inequalities (2.4) that

$$g_1\geqq\frac{r_1|a_3|}{|1+a_3+a_4|},\qquad h_1\geqq\frac{r_2|a_4|}{|1+a_4+a_5|},$$

$$(1-g_{n-1})g_n\geqq\left|\frac{a_{2n}a_{2n+1}}{(1+a_{2n-1}+a_{2n})(1+a_{2n+1}+a_{2n+2})}\right|,$$

$$(1-h_{n-1})h_n\geqq\left|\frac{a_{2n+1}a_{2n+2}}{(1+a_{2n}+a_{2n+1})(1+a_{2n+2}+a_{2n+3})}\right|.$$

If then we put

$$\frac{-a_3a_4}{(1+a_2+a_3)(1+a_4+a_5)}=h_1y_1,\qquad \frac{-a_2a_3}{(1+a_2)(1+a_3+a_4)}=g_1x_1,$$

[10] E. B. Van Vleck, Transactions of this Society, vol. 2 (1901), pp. 476–483.

$$\frac{-a_{2n+1}a_{2n+2}}{(1+a_{2n}+a_{2n+1})(1+a_{2n+2}+a_{2n+3})} = (1-h_{n-1})h_n y_n,$$

$$\frac{-a_{2n}a_{2n+1}}{(1+a_{2n-1}+a_{2n})(1+a_{2n+1}+a_{2n+2})} = (1-g_{n-1})g_n x_n,$$

we see on referring to (1.5), (1.6) that the theorem is true.

Now, both the continued fractions (2.8), (2.9) are convergent. Hence, to prove (1.1) convergent it is required to show that the values of (2.8) and (2.9) are equal. For that purpose we find that[11]

$$\left| \frac{A_n}{B_n} - \frac{A_{n+1}}{B_{n+1}} \right| = \frac{1}{|Q_n Q_{n+1}|}$$

where

$$r_1 r_3 \cdots r_{2n+1} |Q_{2n+2}| \geqq e_1 \left(1 + \sum_{k=1}^{n} r_1 r_3^2 \cdots r_{2k-1}^2 r_{2k+1} |b_{2k+2}|\right),$$

$$r_2 r_4 \cdots r_{2n+2} |Q_{2n+3}| \geqq e_2 \left(1 + \sum_{k=1}^{n+1} r_2 r_4^2 \cdots r_{2k-2}^2 r_{2k} |b_{2k+1}|\right),$$

e_1, e_2 being positive constants, $b_1 = 1$, $a_n = 1/b_{n-1}b_n$ (cf. (1.4)). It is now easy to see that (1.1) converges, under certain conditions, even if the majorant series $1 + \sum r_1 r_2 \cdots r_n$ diverges. It suffices, for example, to have simply $\liminf (r_1 r_2 \cdots r_n) = 0$ provided actual inequality holds for $n = 1, 2$ in (2.4). Other interesting cases are cited in the following theorem.

THEOREM 2.2. *If* (2.4) *holds with actual inequality for* $n = 1, 2$, *then* (1.1) *converges if any one of the following conditions holds:*

(a) $|a_n| < M$, $n = 2, 3, 4, \cdots$, (*M a positive constant*);

(b) $\liminf |a_n| = 0$;

(c) $\liminf (r_1 r_2 \cdots r_n) < \infty$, $\sum (1/|a_n|)$ *diverges*;

(d) $r_1 r_3 \cdots r_{2n-1} < M$, $r_2 r_4 \cdots r_{2n} < M$ (*M a positive constant*), $\sum |b_n|$ *diverges; or*

(e) $\liminf (r_1 r_2 \cdots r_n) < \infty$, $r_1 r_3 \cdots r_{2n-1}$ *and* $r_2 r_4 \cdots r_{2n}$ *bounded away from zero*, $\sum |b_n|$ *diverges.*

3. **The parabola theorem.** As noted in §1, the domain $|a_n| \leqq 1/4$ is the best possible circular convergence region with center at the origin. Perhaps the most interesting result of our investigations of the convergence problem is the theorem which follows.[12]

[11] CT, p. 164.
[12] CT, p. 166.

THEOREM 3.1. *A set of points W which is symmetrical with respect to the real axis is a convergence region*[13] *for the continued fraction* (1.1) *if and only if W is bounded, and is contained in or upon the parabola*

$$|z| - R(z) = \tfrac{1}{2}. \tag{3.1}$$

The necessity of the conditions follows from the fact that if z lies outside the parabola (3.1), then

$$\frac{1}{1} \, + \, \frac{z}{1} \, + \, \frac{\bar{z}}{1} \, + \, \frac{z}{1} \, + \, \frac{\bar{z}}{1} \, + \cdots \tag{3.2}$$

diverges; and from the fact that a convergence region must be bounded, for otherwise $a_2, a_3, a_4, \cdots$ could be chosen in the region in such a way that the series $\sum |b_n|$ of (1.4) would be convergent. The sufficiency follows immediately from Theorem 2.2 (a), since it is easy to verify that the inequalities (2.4) hold with $r_n = 1$, and with actual inequality for $n = 1, 2$, when the a_n's lie in or upon the parabola (3.1).

From Theorem 2.2 (d) we have further that if the a_n's lie in or upon the parabola[14] (3.1), then (1.1) converges if (and only if) the series $\sum |b_n|$ diverges.

No less important and interesting than the convergence region problem is the companion value region problem:[15] if U is a convergence region, to determine a value region V in which the value of (1.1) must lie when $a_2, a_3, a_4, \cdots$ lie in U. If it is known that

(a) u in U implies that $1/(1+u)$ is in V;

(b) u in U, v in V implies that $1/(1+uv)$ is in V,

then it is plain that the approximants A_n/B_n, $(n = 2, 3, 4, \cdots)$, all lie in V whenever $a_2, a_3, a_4, \cdots$ lie in U. By employing this interpretation of the continued fraction as a succession of linear transformations we have succeeded in determining the exact region V in which the approximants of (1.1) must lie when $a_2, a_3, a_4, \cdots$ lie in the parabola (3.1). We find that V is the region

$$|z - 1| \leqq 1, \qquad z \neq 0. \tag{3.3}$$

Every value z satisfying (3.3) is assumed by some continued fraction of the form (1.1) having its partial numerators in the parabola (3.1); and no value z outside this region is so assumed.

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[13] I.e. the continued fraction converges when $a_2, a_3, a_4, \cdots$ have arbitrary values in W.

[14] Not necessarily a bounded portion of the parabola.

[15] W. T. Scott and H. S. Wall, *Value regions for continued fractions* (to be referred to as VR), to be published in this Bulletin.

We have also investigated the value region problem corresponding to a circular convergence region with center at an arbitrary point a not real and $\leqq -1/4$.

It is interesting to consider the following "restricted" value region problem. Suppose that $a_2, a_3, \cdots, a_n$ are fixed in U. To determine a value region V in which the approximants must lie when $a_{n+1}, a_{n+2}, a_{n+3}, \cdots$ vary independently in U. We have considered this problem for the case that U is the region bounded by the parabola (3.1). To illustrate the sort of result obtained, let $a_2=\frac{1}{2}$, $a_3=\frac{1}{6}$, $a_4=\frac{1}{3}$, so that the continued fraction starts out like the simple continued fraction for log 2. Then if (1.1) converges, and $a_5, a_6, a_7, \cdots$ lie in the parabola (3.1), the value of the continued fraction must lie in a circle with center[16] $z=.69375$ (log $2=.69315\cdots$) and radius .00625.

It is interesting to apply these considerations to special continued fractions. For instance, one may show in this way that the hypergeometric function $F(\alpha, 1, \gamma, x)$, (α, γ, real, $0<\alpha<\gamma$), satisfies the inequality[17]

$$\left| F(\alpha, 1, \gamma+1, x) - \frac{\gamma^2}{\gamma^2-\alpha^2} \right| \leqq \frac{\alpha\gamma}{\gamma^2-\alpha^2}, \qquad |x| \leqq 1.$$

4. **The moment problem[18] for the interval** (0, 1). We shall return now to the continued fraction (2.7), in which we shall put $x_n=x$. We shall also multiply by a positive constant factor c_0. We then have:

$$\frac{c_0}{1} + \frac{g_1x}{1} + \frac{(1-g_1)g_2x}{1} + \frac{(1-g_2)g_3x}{1} + \cdots, \tag{4.1}$$

[16] Formulas for the center and radius in terms of a_2, a_3, a_4 are given in VR.

[17] H. S. Wall, *A class of functions bounded in the unit circle*, Duke Mathematical Journal, vol. 7 (1940), pp. 146–153.

[18] By the "moment problem" is ordinarily understood the problem of determining a monotone nondecreasing function $\phi(u)$ satisfying the equations $c_n=\int_a^b u^n d\phi(u)$, $(n=0, 1, 2, \cdots)$, when $\{c_n\}$ is a given sequence. This problem was solved by Stieltjes for the case $a=0$, $b=+\infty$ in 1894 (*Oeuvres*, vol. 2, pp. 402–566). In 1903 E. B. Van Vleck (*On an extension of the 1894 memoir of Stieltjes*, Transactions of this Society, vol. 4 (1903), pp. 297–332) investigated the problem for the case $a=-\infty$, $b=+\infty$. He did not obtain a complete solution. H. Hamburger (Mathematische Annalen, vol. 81 (1920), pp. 235–319, and vol. 82 (1921), pp. 120–164, 168–187) obtained a complete solution in the general case. At about the same time, E. Hellinger (Mathematische Annalen, vol. 86 (1922), pp. 18–29) gave a complete solution in a twelve-page article, by resolving the problem into a question of solving a system of infinitely many linear equations. Another solution was given by T. Carleman, *Sur les équations intégrales singulières à noyau symétrique*, Uppsala, 1923, pp. 189–220.

with $c_0>0$, $0<g_n<1$, $(n=1, 2, 3, \cdots)$. This is a *Stieltjes* type continued fraction inasmuch as the constant factors in the partial numerators are all positive. Since this continued fraction converges uniformly for $|x| \leqq c$ for every positive $c<1$, it follows that the corresponding Stieltjes integral has the form

$$\int_0^1 \frac{d\phi(u)}{1+xu}, \tag{4.2}$$

in which $\phi(u)$ is bounded and monotone nondecreasing and has infinitely many points of increase, $0 \leqq u \leqq 1$. Hence, if $P(x)=\sum c_n(-x)^n$ is the corresponding Stieltjes series, it follows that the sequence $\{c_n\}$ is a *totally monotone sequence*, i.e.,

$$\Delta^m c_n = \int_0^1 (1-u)^m u^n d\phi(u) \geqq 0, \qquad m, n = 0, 1, 2, \cdots .$$

Conversely, let $\{c_n\}$ be a totally monotone sequence. Then, as Hausdorff showed, there exists a function $\phi(u)$ such that

$$c_n = \int_0^1 u^n d\phi(u), \qquad n = 0, 1, 2, \cdots ,$$

where $\phi(u)$ is bounded and monotone nondecreasing. If $\phi(u)$ has infinitely many points of increase ("there is an infinite distribution of mass"), the series $P(x)=\sum c_n(-x)^n$ has a corresponding Stieltjes continued fraction of the form

$$\frac{c_0}{1} + \frac{a_2 x}{1} + \frac{a_3 x}{1} + \cdots, \tag{4.3}$$

in which $a_2, a_3, a_4, \cdots$ are real and positive. Then one may show[19] that (4.3) must be of the form (4.1). Thus, in order for the real sequence $\{c_n\}$ to be totally monotone and correspond to an infinite distribution of mass it is necessary and sufficient that the corresponding Stieltjes continued fraction be of the form (4.1).

As for totally monotone sequences corresponding to a finite distribution of mass, they are completely characterized by having a corresponding continued fraction of the form (4.1) which terminates. In this case the last g_n may be 0 or 1.

The Stieltjes moment problem for the interval (0, 1), for which we have completely characterized the corresponding continued fraction,

[19] H. S. Wall, *Continued fractions and totally monotone sequences* (to be referred to as TM), Transactions of this Society, vol. 48 (1940), pp. 165–184.

is of particular importance on account of its relation to Hausdorff methods of summation. I shall discuss some of our work on this theory in a later section.

5. **A class of real functions bounded in the unit circle.** Let $f(x)$ denote the moment generating function represented by (4.1). Then $f(x)$ has an integral representation of the form (4.2), and a power series expansion $f(x)=\sum c_n(-x)^n$ in which the sequence of coefficients, $\{c_n\}$, is totally monotone. We shall include in our discussion the case where (4.1) terminates, in which event the last g_n may be 0 or 1.

Let $M(f)=\text{l.u.b.}_{|x|<1}|f(x)|$. Then $M(f)\leqq 1$ if and only if the continued fraction can be put into the form[20]

$$\frac{g_1}{1+}\ \frac{(1-g_1)g_2x}{1}\ +\ \frac{(1-g_2)g_3x}{1}\ +\ \cdots \tag{5.1}$$

where $0\leqq g_n\leqq 1$, $(n=1, 2, 3, \cdots)$, and we agree that in case equality holds here for some n, then the continued fraction shall terminate with the first identically vanishing partial quotient. Again, $M(f)\leqq 1$ if and only if $\sum c_n\leqq 1$; or, if and only if

$$f(x)=\int_0^1\frac{(1-u)d\phi(u)}{1+xu},$$

where $\phi(u)$ is monotone and $0\leqq\phi(1)-\phi(0)\leqq 1$.

With each function $e(x)$, analytic for $|x|<1$, and with $M(e)\leqq 1$, Schur[21] associated a sequence of numbers $\{\gamma_n\}$ in the following way. Put $e(x)\equiv e_0(x)$,

$$e_{n+1}(x)=\frac{1}{x}\,\frac{\gamma_n-e_n(x)}{1-\bar\gamma_n e_n(x)},\quad \gamma_n=e_n(0),\qquad n=0, 1, 2, \cdots. \tag{5.2}$$

He showed that either (a) $|\gamma_n|<1$, $n=0, 1, 2, \cdots$; or else (b) $|\gamma_n|<1$, $n=0, 1, 2, \cdots, N-1$, $|\gamma_N|=1$, $\gamma_n=0$, $n>N$. Conversely, if $\{\gamma_n\}$ is any sequence satisfying (a) or (b) then there is uniquely determined a function $e(x)$, analytic for $|x|<1$ and with $M(e)\leqq 1$, such that (5.2) holds.

We shall confine our attention to the class E of *real* functions $e(x)$, analytic for $|x|<1$ and with $M(e)\leqq 1$ In this case the sequence $\{\gamma_n\}$ associated with $e(x)$ in accordance with the result of Schur satisfies

[20] TM, p. 179. Naturally the g_n's are not the same in (5.1), (5.2).

[21] I. Schur, *Ueber Potenzreihen, die im Innern des Einheitskreises beschränkt sind*, Journal für die reine und angewandte Mathematik, vol. 147 (1916), pp. 205–232, and vol. 148 (1917), pp. 122–145.

the condition (a) $-1<\gamma_n<+1$, $n=0, 1, 2, \cdots$; or else the condition (b) $-1<\gamma_n<+1$, $n=0, 1, 2, \cdots, N-1$, $|\gamma_N|=1$, $\gamma_n=0$, $n>N$. The subclass consisting of all moment generating functions of E will be denoted by F. If $e(x)$ is in F, then all the functions $e_1(x), e_2(x), e_3(x), \cdots$ given by (5.2) are also in F.

We shall now prove the theorem which follows.

THEOREM 5.1. *There is a one-to-one correspondence between the functions of E and of F such that if $e(x)$ of E corresponds to $f(z)$ of F, then for $|x|<1$ we have*

$$(5.3)\qquad \frac{(1-x)}{2}\,\frac{1-e(x)}{1+xe(x)} = f(z), \qquad z = 4x/(1-x)^2.$$

This may be formulated in terms of monotone functions: to every function $e(x)$ of E there corresponds a monotone function $\phi(u)$ such that $0\leqq\phi(1)-\phi(0)\leqq 1$ and

$$(5.4)\qquad \frac{1-x}{x}\,\frac{1-e(x)}{1+xe(x)} = \int_0^1 \frac{(1-u)d\phi(u)}{1+zu}, \qquad z = 4x/(1-x)^2, \qquad |x|<1.$$

Conversely, if $\phi(u)$ is any monotone function such that $0\leqq\phi(1)-\phi(0)\leqq 1$, then there exists a function $e(x)$ of E such that (5.4) holds.

Let $e_0(x)\equiv e(x), e_1(x), e_2(x), \cdots$ be determined by (5.2), and put

$$p_n(x) = \frac{1-e_n(x)}{1+xe_n(x)}, \qquad n = 0, 1, 2, \cdots.$$

Then from (5.2) we have the relation

$$p_n(x) = \frac{1-\gamma_n}{1-x+(1+\gamma_n)xp_{n+1}(x)}, \qquad n = 0, 1, 2, \cdots,$$

and consequently there is the *formal* continued fraction expansion

$$(5.5)\quad \tfrac{1}{2}(1-x)\frac{1-e(x)}{1+xe(x)} = \frac{g_1}{1\,+}\;\frac{(1-g_1)g_2z}{1\qquad+}\;\frac{(1-g_2)g_3z}{1\qquad+}\;\cdots,$$

where $z=4x/(1-x)^2$, $g_n=\frac{1}{2}(1-\gamma_{n-1})$, $(n=1, 2, 3, \cdots)$. If some γ_n is $+1$ or -1, this continued fraction terminates and (5.5) is then an identity. If, on the other hand, $-1<\gamma_{n-1}<+1$, then $0<g_n<1$, and the continued fraction converges uniformly for $|z|\leqq 1$, or for x in a sufficiently small neighborhood of the origin. One may easily show that the power series expansion in ascending powers of x of the function on the left agrees term-by-term with the power series expansion in ascending powers of x of the nth approximant of the continued fraction, for more and more terms as n is increased. From these facts one may conclude[22] that (5.5) is a true equality for x in a sufficiently small neighborhood of the origin. Now,[23] the function $z=4x/(1-x)^2$ maps the interior of the circle $|x|=1$ in a one-to-one manner upon the z-plane exterior to the cut along the real axis from -1 to $-\infty$. The continued fraction represents an analytic function of z in this cut plane, and therefore represents an analytic function of x for $|x|<1$. Thus (5.5) is a true equality for $|x|<1$.

As previously pointed out, this continued fraction represents a function $f(z)$ of the class F.

Conversely, starting with the function $f(z)$ of F, so that the numbers g_n may be found, we then determine the sequence $\{\gamma_n\}$ and hence the function $e(x)$ of E, by putting $\gamma_{n-1}=1-2g_n$, $(n=1, 2, 3, \cdots)$. If $0<g_n<1$, then $-1<\gamma_n<+1$, and the g_n's and γ_n's are uniquely determined. In case the continued fraction for $f(z)$ terminates, we may always assume that the last g_n is either 0 or 1, and take subsequent g_n's all equal to $\frac{1}{2}$. In this case we determine $e(x)$ as before by taking $\gamma_{n-1}=1-2g_n$. The function $e(x)$ clearly satisfies (5.5), and the theorem is proved.

Denote by $q(x)$ the function in the left member of (5.5). Then we find that

$$\frac{1-q(x)}{1+zq(x)}=\frac{1}{2}(1-x)\frac{1+e(x)}{1-xe(x)}\cdot$$

Now if we replace $e(x)$ by $-e(x)$ in (5.5), it is clear that the effect upon the continued fraction on the right is to replace γ_{n-1} by $-\gamma_{n-1}$, which is the same as replacing g_n by $1-g_n$. Hence we have the theorem[24] that if

$$p(x)=\frac{g_1}{1}+\frac{(1-g_1)g_2x}{1}+\frac{(1-g_2)g_3x}{1}+\cdots,$$

then the continued fraction for $[1-p(x)]/[1+xp(x)]$ is obtained

[22] Cf. the argument in Perron, p. 343.

[23] Cf. E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin, 1929, p. 112.

[24] This theorem also follows from Theorem 2.1, p. 166, of TM. This was used in the paper referred to in Footnotes 14, 24 and also in the paper by H. S. Wall: *A continued fraction related to some partition formulas of Euler*, American Mathematical Monthly, vol. 48 (1941), pp. 102–108.

from that for $p(x)$ by replacing g_n by $1-g_n$, $(n=1, 2, 3, \cdots)$. It follows that if $p(x)$ is in the class F, so is $[1-p(x)]/[1+xp(x)]$. Therefore we have, in consequence of Theorem 5.1, the following theorem:

THEOREM 5.2. *To each function $p(x)$ of the class F there corresponds uniquely a function $f(z)$ of F such that*

$$(1-x)p(x) = 2f(z), \qquad z = 4x/(1-x)^2, \quad |x| < 1. \tag{5.6}$$

From Theorem 5.1 we obtain the following result:

THEOREM 5.3. *Let $e(x)$ be a function of E such that $-1<\gamma_n<+1$, $(n=0, 1, 2, \cdots)$. Let the series*

$$S = 1 + \sum_{n=0}^{\infty} \frac{(1-\gamma_0)(1-\gamma_1)\cdots(1-\gamma_n)}{(1+\gamma_0)(1+\gamma_1)\cdots(1+\gamma_n)}$$

be convergent. Then $M(e)=1$.

It is interesting to compare this with the result of Schur which states that if the series $\sum|\gamma_n|$ converges, then $M(e)<1$.

To prove the theorem, let us recall that the series S is the series appearing in (2.6) with $g_n=\frac{1}{2}(1-\gamma_{n-1})$. Hence, when this series converges we conclude by Theorem 5.1 that

$$|q(x)| = \left| \frac{1-x}{2} \, \frac{1-e(x)}{1+xe(x)} \right| \leqq (1-1/S) < 1$$

if $|4x/(1-x)^2|<1$. Let x be real, and let $x\to -1$, $|x|<1$. Then $|4x/(1-x)^2|<1$ and $4x/(1-x)^2\to -1$, so that $q(x)\to 1-(1/S)$. It follows that $e(x)\to 1$, so that $M(e)=1$.

Inasmuch as $M[e(x)]=M[-e(x)]=M[e(-x)]=M[-e(-x)]$, it follows that the convergence of one of the series obtained from S by replacing γ_n by $-\gamma_n$, γ_{2n+1} by $-\gamma_{2n+1}$, or γ_{2n} by $-\gamma_{2n}$, $(n=0, 1, 2, \cdots)$, is sufficient in order that $M(e)=1$.

By a result of Stieltjes, the function $f(z)$ represented by the continued fraction on the right of (5.5) is a meromorphic function of z if and only if $\lim_{n=\infty}(1-g_n)g_{n+1}=0$, that is:

$$\lim_{n=\infty}(1+\gamma_{n-1})(1-\gamma_n) = 0.$$

In this case we may write

$$q(x) = \frac{(1-x)[1-e(x)]}{2[1+xe(x)]} = (1-x)^2 \sum_{n=1}^{\infty} \frac{M_n}{(x-r_n)(x-\bar{r}_n)}$$

where $M_n>0$, $(n=1, 2, 3, \cdots)$, $\sum M_n$ converges, and $r_n=1-2u_n+2i[u_n(1-u_n)]^{1/2}$, $1>u_1>u_2>\cdots>0$, $\lim u_n=0$. Thus the singularities of $q(x)$ consist of simple poles r_n, $\bar{r}_n$ lying upon the circle $|x|=1$, together with the point $x=1$ which is the unique limit point of these poles. The function $q(x)$ satisfies the relation $q(1/x)=q(x)$. The following theorem is now evident.

THEOREM 5.4. *If $e(x)$ is a function of E such that $-1<\gamma_n<+1$, $\lim_{n=\infty}(1+\gamma_{n-1})(1-\gamma_n)=0$, then $e(x)$ has an essential singularity at $x=1$, and no other singularities in the extended plane except poles. The function*

$$q(x) = \frac{1-x}{2}\,\frac{1-e(x)}{1+xe(x)}$$

has as its singularities an infinite sequence of simple poles lying on the circle $|x|=1$ with the single limit point $x=1$, and no others; and $q(1/x)=q(x)$.

As an example, if

$$e(x) = -\frac{1}{x}\,\frac{\dfrac{2x^{1/2}}{1+x} - \tanh\dfrac{2x^{1/2}}{1-x}}{\dfrac{2x^{1/2}}{1+x} + \tanh\dfrac{2x^{1/2}}{1-x}},$$

then $q(x)=3^{-1}/1+3^{-1}5^{-1}z/1+5^{-1}7^{-1}z/1+\cdots$, $z=4x/(1-x)^2$, as one may verify by means of the continued fraction of Lambert for $\tanh z$. This function $e(x)$ satisfies the conditions of Theorem 5.4. When $z=-1$ in the continued fraction for $q(x)$, we find that its value is $1-(\tan 1)^{-1}<1$, and consequently the series S of Theorem 5.3 converges. From that theorem it then follows that $M(e)=1$. This can be verified directly by letting x approach -1 in the above expression for $e(x)$.

Theorem 5.1 admits of the following interpretation. In (5.3) put $[1-e(x)]/[1+xe(x)]=c_0-c_1x+c_2x^2-\cdots$, and $f(z)=C_0-C_1z+C_2z^2-\cdots$. Considered as an identity in x, (5.3) yields the equations[25]

$$(5.7)\qquad c_n = 2^{2n+3}C_{2n,0}C_n - 2^{2n+1}C_{2n-1,1}C_{n-1} + \cdots + (-1)^n 2^3 C_{n,n}C_0,$$

which constitute a linear transformation of the sequence $\{C_n\}$ into

[25] W. T. Scott and H. S. Wall, *Linear manifolds of Hausdorff means* (to be referred to as LM). To appear in the Transactions of this Society as Part I of a paper: *The transformation of series and sequences.*

the sequence $\{c_n\}$. The inverse transformation is found to be

$$(5.8)\qquad \begin{aligned} C_n &= \frac{1}{(n+1)2^{2n+1}} C_{2n,n}c_0 + \frac{3}{(n+2)2^{2n+1}} C_{2n,n-1}c_1 + \cdots \\ &\quad + \frac{2n+1}{(2n+1)2^{2n+1}} C_{2n,0}c_n. \end{aligned}$$

The transformation (5.8) has the property that it carries any sequence $\{c_n\}$ generated by a function $[1-e(x)]/[1+xe(x)]$ where $e(x)$ is in E, into a totally monotone sequence $\{C_n\}$ with sum $\sum C_n \leqq 1$; and (5.7) has the property that it carries any totally monotone sequence $\{C_n\}$ with sum $\leqq 1$ into a sequence $\{c_n\}$ which is generated by a function of the form $[1-e(x)]/[1+xe(x)]$ where $e(x)$ is in E. This leads us to the topic to be discussed in the next section.

6. **Hausdorff means.** The familiar $(C, 1)$ transform of a sequence $\{s_n\}$, namely: $t_m = (s_0+s_1+\cdots+s_m)/(m+1)$, may be written in the form

$$t_m = \sum_{n=0}^{m} C_{m,n} \int_0^1 (1-u)^{m-n}u^n s_n du.$$

One may generalize this by replacing du by $d\phi(u)$ where $\phi(u)$ is any function of bounded variation on the interval $0 \leqq u \leqq 1$, which is continuous at $u=0$, and which satisfies the conditions $\phi(0)=0$, $\phi(1)=1$. The resulting transform of the sequence $\{s_n\}$ is the Hausdorff mean.[26] This defines a regular method of summation, i.e., $s_n \rightarrow s$ implies $t_m \rightarrow s$, which is denoted by the symbol $[H, \phi(u)]$.

If we put $c_n = \int_0^1 u^n d\phi(u)$, $(n=0, 1, 2, \cdots)$, the Hausdorff mean can be written in the form

$$t_m = \sum_{n=0}^{m} C_{m,n}\Delta^{m-n}c_n \cdot s_n.$$

The function $\phi(u)$, subject to the conditions imposed above, is called a *regular mass function*; and the sequence $\{c_n\}$ is called a *regular moment sequence*. Included among the Hausdorff means are the Hölder means (H, α), the Cesàro means (C, α), and many others.

Let $[H, \phi_a(u)]$ and $[H, \phi_b(u)]$ be two Hausdorff methods of summation, and let $\{a_n\}$ and $\{b_n\}$ be the corresponding regular moment se-

[26] F. Hausdorff, *Summationsmethoden und Momentfolgen*, I and II, Mathematische Zeitschrift, vol. 9 (1921), pp. 74–109, 280–299. For an elementary account, see H. L. Garabedian, *Hausdorff matrices*, American Mathematical Monthly, vol. 46 (1939), pp. 390–410.

quences. Then[27] $[H, \phi_a(u)] \supset [H, \phi_b(u)]$, when $b_n \neq 0$, $(n = 0, 1, 2, \cdots)$, if and only if there exists a regular moment sequence $\{c_n\}$ such that

$$a_n = b_n c_n, \qquad n = 0, 1, 2, \cdots. \tag{6.1}$$

This is the basic theorem of the Hausdorff theory. When (6.1) holds it is convenient to say that $\{a_n\}$ is *divisible* by $\{b_n\}$. The condition that $\{a_n\}$ be divisible by $\{b_n\}$ may be formulated in a number of different ways.[28] One of the most natural ways is in terms of the moment generating functions

$$f_a(x) = \sum a_n(-x)^n, \qquad f_b(x) = \sum b_n(-x)^n.$$

The regular moment sequence $\{a_n\}$ is divisible by the regular moment sequence $\{b_n\}$ if and only if there exists a regular mass function $\phi_c(u)$ such that

$$f_a(x) = \int_0^1 f_b(ux) d\phi_c(u)$$

for all x not real and $\leqq -1$. By means of the Stieltjes inversion formula one may show that this holds if and only if

$$\phi_a(u) = \phi_c(u) + \int_u^1 \phi_b(u/v) d\phi_c(v)$$

for all except at most a countable set of values of u.

If $\phi(u)$ is monotone nondecreasing, then $\{c_n\}$ is a totally monotone sequence, and hence the moment generating function $f_c(x)$ has a corresponding continued fraction of the form (4.1). It is easy to obtain conditions on (4.1) which are necessary and sufficient for $\{c_n\}$ to be a regular moment sequence.[29] An interesting example is afforded by the continued fraction of Gauss for the hypergeometric function $F(\alpha, 1, \gamma, x)$, α, γ real, $0 < \alpha < \gamma$. We have investigated in some detail the "hypergeometric summability" defined in this way.

It is interesting that in some cases one may operate directly with the continued fraction (4.1) to prove inclusion relationships between Hausdorff methods.[30]

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[27] This means that every sequence summable $[H, \phi_b(u)]$ is summable $[H, \phi_a(u)]$, and is read "$[H, \phi_a(u)]$ includes $[H, \phi_b(u)]$."

[28] H. L. Garabedian, Einar Hille, and H. S. Wall, *Formulations of the Hausdorff inclusion problem*, Duke Mathematical Journal, vol. 8 (1941), pp. 193–213.

[29] H. L. Garabedian and H. S. Wall, *Hausdorff methods of summation and continued fractions*, Transactions of this Society, vol. 48 (1940), pp. 185–207, p. 188.

[30] H. L. Garabedian and H. S. Wall, *Continued fractions and Hausdorff methods of summation*, Northwestern University Studies, in press.

Beginning with the interpretation of Theorem 5.1 given at the end of §5, we have developed a theory of linear manifolds of Hausdorff means.[31] Returning to (5.8), let us put

$$\beta_n(u) = \frac{1}{2^{2n+1}}\left\{\frac{1}{n+1}C_{2n,n} + \frac{3}{n+2}C_{2n,n-1}u + \cdots + \frac{2n+1}{2n+1}C_{2n,0}u^n\right\}.$$

Then if $c_n=\int_0^1 u^n d\phi(u)$, we may write the transformation (5.8) as

$$C_n = \int_0^1 \beta_n(u)d\phi(u), \qquad n = 0, 1, 2, \cdots. \tag{6.2}$$

From the discussion in §5 it follows that if $\{c_n\}$ is totally monotone and[32] $\sum c_n \leqq 1$, the sequence $\{C_n\}$ is of like character. This is, however, only part of the story. We find that if $\phi(u)$ is any function of bounded variation on the interval $0\leqq u\leqq 1$, such that $\phi(1)-\phi(0)=1$, then the sequence $\{C_n\}$ is a regular moment sequence.

Let $M(\beta_n(u))=M(\beta)$ denote the set of all moment sequences $\{C_n\}$ obtained from (6.2) by letting $\phi(u)$ run through the class of all functions of bounded variation on the interval $0\leqq u\leqq 1$. It is observed that if $\{C_n'\}$ and $\{C_n''\}$ are any two sequences in $M(\beta)$ then the sequences $\{C_n'+C_n''\}$, and $\{KC_n'\}$, (K a constant), are in $M(\beta)$. Thus $M(\beta)$ is a *linear manifold* of moment sequences.

Any suitably chosen sequence of functions $\{\beta_n(u)\}$ determines in this same way a linear manifold of moment sequences. We have called $\{\beta_n(u)\}$ the *basis* of the manifold. In laying down the outline for a general theory of these manifolds, we have obtained conditions on a sequence of functions $\{\beta_n(u)\}$ in order that it form the basis of a manifold; have obtained conditions under which the sequences of the manifold are all essentially regular; and conditions under which the Hausdorff methods defined by the sequences of a manifold all include a given Hausdorff method.

As an example, let $\beta_n(u)=(u+1)/(u+n+1)$. For every fixed u in the interval $0\leqq u\leqq 1$, the sequence $\{\beta_n(u)\}$ is totally monotone. From this one may conclude that $\{\beta_n(u)\}$ is a basis for a manifold $M(\beta)$. Further, it can be shown that every sequence $\{C_n\}$ in $M(\beta)$ for which $C_0\neq 0$ is essentially regular; and that every Hausdorff method of summation defined by the sequences of $M(\beta)$ includes $(C, 1)$ but not all of them include $(C, 1+t)$, $t>0$.

7. **Continued fraction expansions for arbitrary power series.** The

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[31] LM.

[32] If $f(x)$ is in F and $f(x)=\sum c_n(-x)^m$, then $M(f)=\sum c_n$. Cf. TM, p. 181.

continued fraction expansions for power series which we have used up to this point all have partial quotients of the form $a_n x/1$. This is true only when the power series is of a special kind. Any power series $P(x)=1+c_1x+c_2x^2+\cdots$ has a corresponding continued fraction in which the partial quotients are of the form $a_n x^{\alpha_n}/1$, α_n a positive integer.[33] The continued fraction has many of the properties of semi-normal continued fractions. For example, it terminates if and only if $P(x)$ represents a rational function of x; and if the continued fraction converges uniformly in the vicinity of the origin, the power series also converges in the neighborhood of the origin and the power series is equal to its continued fraction.

We have investigated these "corresponding type" continued fractions with a view toward obtaining formulas for the a_n's and α_n's in terms of the power series.[34]

We found first of all that there is a fairly practical step-by-step process for expanding $P(x)$ into a continued fraction. This is based upon the observation that if we have carried out the expansion to the point where we have

$$1+\frac{a_1x^{\alpha_1}}{1}+\frac{a_2x^{\alpha_2}}{1}+\cdots+\frac{a_nx^{\alpha_n}}{1},$$

and we denote this fraction by $A_n(x)/B_n(x)$, then the formal power series for

$$\frac{A_n(x)}{B_n(x)}-P(x) \tag{7.1}$$

must begin with the term $(-1)^{n+1}a_1a_2\cdots a_{n+1}x^{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}}$. Hence, knowing $a_1, a_2, \cdots, a_n, \alpha_1, \alpha_2, \cdots, \alpha_n$ we may determine a_{n+1} and α_{n+1} as soon as we know the term of lowest degree in the power series for (7.1). This process requires expansions in powers of x for only a *finite* number of terms at each stage.

We have found formulas for the elements of the continued fraction in terms of the power series in the case of a large class of continued fractions which we have termed *absolutely regular*, namely those for which $\alpha_1=1$ and

[33] Walter Leighton and W. T. Scott, *A general continued fraction expansion*, Bulletin of this Society, vol. 45 (1939), pp. 596–605.

[34] W. T. Scott and H. S. Wall, *Continued fraction expansions for arbitrary power series*, Annals of Mathematics, (2), vol. 41 (1940), pp. 328–349.

$$\begin{aligned} \alpha_2 + \alpha_4 + \cdots + \alpha_{2n} &\geqq \alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1}, \\ \alpha_3 + \alpha_5 + \cdots + \alpha_{2n+1} &\geqq \alpha_2 + \alpha_4 + \cdots + \alpha_{2n}, \end{aligned} \qquad n = 1, 2, 3, \cdots .$$

In case the a_n's are real and positive, these conditions are equivalent to the condition that all the approximants of the continued fraction be Padé approximants for the corresponding power series. The class of absolutely regular continued fractions contains the class of semi-normal continued fractions, and the formulas found are natural extensions of the well known formulas for the semi-normal case.

A remarkable situation arises when each exponent in the power series $P(x)$ is at least twice the preceding. In this case the a_n's in the continued fraction depend only upon the coefficients in the power series, while the α_n's in the continued fraction depend only upon the exponents in the power series. For this reason, certain operations upon the power series and continued fraction, such as differentiation, integration, forming reciprocals, and a sort of Hadamard composition, may be readily performed.

There is considerable evidence to support the conjecture that corresponding type continued fractions represent functions having circles as natural boundaries, provided the exponents α_n increase sufficiently rapidly. We have found that this is the case when the a_n's are suitably restricted and the α_n's form a geometric progression.

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