214. C. D. Olds: *On some arithmetical identities.*

In this note the author considers certain arithmetical identities of the type: $\sum \cos [2(a+b)\pi/p] = \sum \cos [2(a-b)\pi/p]$, $b \neq a$, where a and b range successively over all the quadratic residues of the given prime number p. Some applications are indicated. (Received March 8, 1941.)

215. Max Zorn: *Transcendental p-adic numbers related to roots of unity.*

Elementary remarks about some sequences which converge p-adically. (Received March 10, 1941.)

Analysis

216. E. F. Beckenbach: *On functions having subharmonic logarithms.*

Two new characterizations of functions having subharmonic logarithms are given. One is expressed in function-theoretic terms, while the other is given by a generalized isoperimetric inequality. (Received March 10, 1941.)

217. Stefan Bergman: *The method of the minimum integral and the analytic continuation of functions.*

The measures of geometrical objects which occur in the theory of CT's (conformal transformations) and PT's (pseudo-conformal transformations) can be expressed as functions of minima $\lambda_{\mathfrak{B}}(t)$ of integrals $\int_{\mathfrak{B}} |h|^2 dx_1 dy_1 \cdots dx_n dy_n$, $z_k = x_k + iy_k$, where h runs through all functions analytic in $\mathfrak{B}$ and subjected to certain conditions at the point (t). If $\mathfrak{G} \subset \mathfrak{B}$ then $\lambda_{\mathfrak{G}}(t) \leqq \lambda_{\mathfrak{B}}(t)$, and since the equation $ds^2_{\mathfrak{B}}(z) = |dz|^2 / \lambda_{\mathfrak{B}}(t)$ defines a metric invariant with respect to CT's the lemma of Schwarz-Pick (see Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 16 (1937), p. 11) is obtained. On the other hand $\lambda_{\mathfrak{B}}(t)$ can be expressed with the aid of a system of functions orthogonal in $\mathfrak{B}$. Thus, in the case of Schlicht CT's one obtains certain refinements of the lemma of Schwarz-Pick. The introduction of the concept of B-area enables the author to generalize this technique to PT's. Finally, if one supposes that functions $w_k(z_1, z_2)$, $k=1, 2$, of the PT satisfy certain conditions on a surface bounding a segment $\mathfrak{a}$ of the boundary then w_1, w_2 are regular in $\mathfrak{a}$, and the PT (w_1, w_2) can be extended analytically through $\mathfrak{a}$ outside its original domain of definition. Applying then the method of the minimum integral for the extended domain, the author obtains results concerning distortion on the boundary for PT's. (Received April 1, 1941.)

218. Lipman Bers: *On a generalized harmonic measure.*

Suppose D is a domain in R_n, B its (bounded) boundary, B' the set of the regular boundary points of D, $u(P)$ $(P \in D)$ a bounded harmonic function (b.h.f.). There exists a function $\mu_u(P, e)$ (generalized harmonic measure) which is a b.h.f. of P for every fixed Borelian set $e \subset B$, a completely additive set function for every fixed point $P \in D$ and satisfies the condition $\lim [\mu_u(P, e) - u(P)] = 0$, $\lim \mu_u(P, B-e) = 0$ $(P \to R \in B')$ for every open set $e \subset B$, $R \in e$. If $f(Q)$ is a continuous function, $v(P) = \int_B f(Q) d\mu_u(P, e_Q)$ is the solution of the following problem (a generalization of the Dirichlet problem): determine a b.h.f. v such that $\lim [v(P) - f(R)u(P)] = 0$ $(P \to R \in B')$. If f is a bounded Baire function, v is a b.h.f. which is "representable by u." The bounded harmonic functions which possess a positive lower bound and which are representable by a given function u form a "representation class" $C(u)$. Every function of $C(u)$ is representable by any other function of this class and only by these functions. In a given domain there exists either only one or continuously many different representation classes. (Received April 1, 1941.)

219. P. B. Burcham: *Some inclusion relations in the Hausdorff difference matrix.*

The matrix $(\Delta^m c_u)$, where $\{c_n\}$ is a regular Hausdorff moment sequence, has been defined as the difference matrix (Transactions of this Society, vol. 48 (1940), pp. 185–207). In this paper it is established that corresponding row sequences of the matrices $(\Delta^m a_n)$ and $(\Delta^m b_n)$ define equivalent methods of summation provided that the sequences $\{a_n\}$ and $\{b_n\}$ define the equivalent methods of summation, Cesàro summability $(C, \gamma-\alpha)$ and hypergeometric summability $(H, \alpha, 1, \gamma)$, $\Re(\alpha, \gamma, \gamma-\alpha)>0$. Additional results are obtained concerning the column and diagonal sequences of these two matrices. In the case that $\{a_n\}$ and $\{b_n\}$ define the Cesàro methods (C, α) and (C, β), $\Re(\alpha, \beta)>0$, it is found that all of the diagonal sequences of the associated difference matrices define equivalent methods of summation. (Received February 24, 1941.)

220. Richard Courant: *On a new type of variational problems, with physical demonstrations.*

The classical calculus of variations concerns joining extremals between two points, or spanned within a closed curve, or problems of a similar type. There are, however, problems of equilibrium in physics that suggest a more general type of variational problems, of which the following is an example: Given n points in the plane; find a simply connected graph of extremal arcs joining the n points and having stationary length in the variational metric. A famous classical problem by Steiner gives the simplest illustration. There is a connection between these problems and isoperimetric problems with inequalities as subsidiary conditions whereby in particular any ordinary variational problem appears as a limiting case of an isoperimetric problem. The position and the treatment of these general problems carries over to more dimensions; for example: Given in three dimensions a closed graph (say the system of edges of a cube); desired a system of minimal surfaces bounded by the graph and having common boundary lines to be determined, such that the total area becomes stationary. The problems, their solution, and their interdependence can be illustrated by simple soapfilm experiments. (Received March 31, 1941.)

221. E. L. Crow: *The expansion problem associated with an ordinary differential equation of first order in which the coefficient is a quadratic polynomial in the parameter.*

Consider the differential equation $dy(x, \lambda)/dx=p(x, \lambda)y(x, \lambda)$, where x is a real variable and λ is a complex parameter. The expansion problem associated with this equation and a two-point boundary condition has been discussed by R. E. Langer (Transactions of this Society, vol. 25 (1923), pp. 155–172) for the case in which $p(x, \lambda)$ has a finite number of simple poles as a function of λ. Convergence was established by expressing the series of characteristic solutions as a sum of residues at the characteristic values. However, the method fails when $p(x, \lambda)$ has a double pole, in particular when $p(x, \lambda)$ is of the form $a(x)\lambda^2+b(x)\lambda+c(x)$. The present paper approximates the circumstances of the latter problem by replacing one of the boundary points on the real axis by two neighboring points in the complex plane. The variable x is allowed to range over a simply-connected region including the three boundary points. An adaptation of a method developed by Langer (Transactions of this Society, vol. 46 (1939), pp. 151–190) for use in the complex domain is employed to establish the uniform convergence of simultaneous expansions of two arbitrary functions. (Received March 14, 1941.)

222. R. J. Duffin: *Möbius transforms and Fourier transforms.*

Let $g(x)$ be the Fourier cosine transform of $f(x)$. It is shown that formally $f(x)=(2\pi)^{1/2}\sum_{n=1}^{\infty}\mu_n/nx\sum_{m=1}^{\infty}g(2\pi m/nx)$. Here μ_n is the familiar Möbius symbol having the values 0, 1, -1. The application of suitable summability methods validates this inversion formula for a wide class of functions. It is sufficient to assume that $f(x)$ be absolutely integrable. Conditions may also be imposed on $g(x)$; for instance, the formula is valid if $g(x)$ is of bounded total variation. (Received March 13, 1941.)

223. Benjamin Epstein: *Asymptotic problems associated with a Laplace-Mellin integral equation.* Preliminary report.

In this paper the author studies the integral equation $\psi(s)=\int_A^\infty\sum_{n=0}^{\infty}(\lambda+n)^{-s}f(\lambda)d\lambda$, $A\geqq 0$, Re $s>1$, $f(\lambda)$ L-integrable. In the first half of his paper he proves Abelian and Tauberian theorems for this equation relating the asymptotic properties of $\psi(s)$ as $s\rightarrow 1^+$ with those of $f(\lambda)$ as $\lambda\rightarrow\infty$. The technique utilized is the repeated application of Tauberian and Abelian theorems for the Laplace integral equation. The second half of the paper is devoted to a moment problem analogous to the Stieltjes moment problem over the infinite interval. For example, the following question is considered: Suppose s runs through the monotone non-decreasing set of real numbers σ_n, $n=1, 2, \cdots$, with limit point infinity; then are there conditions on the size of $\psi(\sigma_n)$ and the density of σ_n which assure uniqueness of solution? Analytic criteria for this problem are found. The chief tools utilized are theorems by Carleman and Junnila for functions analytic in a half-plane and a theorem of Boas (Transactions of this Society, vol. 46, pp. 142–150). (Received March 11, 1941.)

224. H. H. Goldstine: *The parametric problem of the calculus of variations in general analysis.*

The well known problem of the calculus of variations is here extended so that the integrand is defined on a region of the spaces of points (t, x_p, r_p), where p ranges over an arbitrary class P. Analogues of the usual necessary conditions, and at least some of the sufficiency conditions, are obtained. The principal point of departure from the results for the problem in finitely many dimensions is in the discussion and character of extremals. (Received March 11, 1941.)

225. W. J. Harrington and J. B. Rosser: *A study of certain functions auxiliary to Brun's method in number theory.*

In 1937 J. B. Rosser made a preliminary report on *An improvement of Brun's method in number theory* (this Bulletin, abstract 43-3-142). Certain upper and lower bounds stated in that report were obtained by approximating certain functions by very extensive computations. In the present paper these functions are studied and much superior methods of evaluation are devised. The functions consist of infinite series, the terms of which are multiple integrals. The principal aim of the paper is to establish certain identities which express the sums of the series in closed form. Preliminary to proving these identities, one has to prove convergence, continuity, and existence of derivatives. These matters take up the bulk of the paper. In the report mentioned above, the results stated were relevant only to a particular application of Brun's method. The functions studied in this paper are taken to be sufficiently general so that the evaluations obtained can be used in the entire range of application of Brun's method. This paper prepares the way for the publication of the paper of which the preliminary report was made in 1937. (Received March 26, 1941.)

226. J. F. Heyda: *A uniqueness property of general monogenic functions.*

Let $f(\alpha)$ be the class of functions defined by $f(\alpha)=\iint q(x, y)(z-\alpha)^{-1}dxdy$ (α in E) integrated over the complement $C(E)$ of a closed set E in a bounded domain K of the α-plane. The function $q(x, y)$ is real and continuous in $\overline{K}$ and vanishes at points of E; moreover $z=x+iy$, where (x, y) is a point in $C(E)$. The form of the functions $f(\alpha)$ is suggested by the non-analytic part of the derivative of $F(\alpha)$, where $F(\alpha)$ is general monogenic in E (as defined by W. J. Trjitzinsky, Annales de l'École Normal Supérieure, vol. 55, pp. 119–191) and is represented there by $F(\alpha)=h(\alpha)+\iint_{C(E)}\log(z-\alpha)q(x, y)dxdy$. Suppose E contains a closed interval I and Γ is a closed linear set of points in I having positive measure. The following uniqueness problem is considered: what must be the "rarefication" of the set $C(E)$, or, alternately, how fast must $|q(x, y)|\to 0$ as $\rho\to 0$ (ρ is the distance of (x, y) from the frontier of $C(E)$) in order that knowledge of the functional values of $f(\alpha)$ on Γ will determine uniquely the values of $f(\alpha)$ on Γ', where $I\supset\Gamma'\supset\Gamma$ and measure $\Gamma'>$measure Γ. (Received February 28, 1941.)

227. Dunham Jackson: *Generalization of a theorem of Korous.*

An elementary treatment of the convergence of series of orthogonal polynomials is greatly facilitated if the polynomials of the orthonormal set are known to be uniformly bounded on the domain of orthogonality, or on a part of it where convergence is to be proved. A demonstration due to J. Korous (see G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, p. 157) shows in a few lines that the orthonormal polynomials corresponding to a weight function $\rho\sigma$ on an interval are thus bounded, if the polynomials for weight ρ have the desired property, and if the factor σ satisfies a Lipschitz condition and has a positive lower bound on the entire domain of orthogonality. The purpose of this note is to show that the argument of Korous can be extended so as to apply under fairly general conditions to orthogonal polynomials in two real variables on an algebraic curve, and in particular to orthogonal trigonometric sums, which can be regarded as orthogonal polynomials on a circle. (Received February 6, 1941.)

228. H. Kober: *On a theorem of Schur and on fractional integrals of purely imaginary order.*

Schur's theorem ideals with linear transformations on L_2 to L_2 involving a kernel $K(x, y)$, homogeneous of degree -1, such that $K(x, 1)x^{-1/2}\in L_1(0, \infty)$. It is shown that two types of fractional integrals of order α belong to this class of transformations for $\Re(\alpha)>0$ and tend to bounded limits in $L_2(0, \infty)$ as $\Re(\alpha)\to 0$. The inversion and group properties of such fractional integrals of purely imaginary order are studied.

The results admit of partial extensions to L_p. Characteristic values and functions are also determined. (Received February 13, 1941.)

229. R. E. Lane: *The first three rational operations.*

This paper establishes the foundation for an approach to the number system in a manner designed to postpone the need for infinite processes such as Dedekind cuts. In contrast to the Cantor and Dedekind approaches, it considers numbers as being generated from the unit-elements of a sequence of rational operations analogous to addition and multiplication. Postulates are laid down for such a sequence of operations, and a number of theorems are derived for the first three rational operations: intersection, union, and addition. (Received March 4, 1941.)

230. Lincoln La Paz: *Double integral variation problems with prescribed transversality coefficients.*

H. A. Simmons has recently published an interesting derivation of the transversality relationship for the variable limit problem of the calculus of variations for n-tuple integrals (Transactions of this Society, vol. 36 (1934), pp. 29–43). It is the purpose of the present note to formulate and solve an inverse problem suggested by this transversality relationship. Attention is restricted to the double integral case but the argument made and the conclusions drawn are easily extended to n-tuple integrals. Transversality coefficients are defined for a regular double integral problem. Necessary and sufficient conditions for a pair of functions to be transversality coefficients of such a problem are obtained and the integrand function of the most general variation problem associated with a pair of transversality coefficients is determined. (Received March 13, 1941.)

231. Hans Lewy: *Fractional potentials.*

In the theory of the logarithmic potential log r, Laplace's formula gives the distribution of mass in terms of the potential. The author derives an analogous explicit formula for potentials $r^{-\beta}$ with β constant and $0<\beta<2$. Under certain rather general conditions, for a given function U, a mass distribution is determined whose potential is U. Similar formulae hold in spaces of arbitrary dimension. They can be utilized in the construction of the conductor potential by reducing this problem to that of an integral equation of second kind. The restriction of β to negative numbers may be dropped. (Received February 25, 1941.)

232. A. T. Lonseth: *Dirichlet's principle for certain nonlinear elliptic equations.*

Courant's "Dirichlet principle" method of solving the Dirichlet problem for self-adjoint linear partial differential equations of elliptic type (*Methoden der Mathematischen Physik*, vol. 2, Berlin, 1937) is extended to nonlinear equations of type $\Delta u = 1/2\ \partial P(x, y; u)/\partial u$, where Δ is the Laplacian differential operator and $P(x, y; u)$ is non-negative, analytic in its three arguments, and convex in u. The (unique) solution automatically minimizes the integral whose first variation it nullifies. An alternative proof of the minimizing property is obtained. Previous existence proofs for this type of nonlinear elliptic equation are due to Bieberbach (method of successive approximations) and to Lichtenstein (method of Ritz). Analyticity of $P(x, y; u)$ may be replaced by less stringent conditions, and Δ by a more general self-adjoint linear elliptic operator. (Received March 12, 1941.)

233. L. H. Loomis: *A converse to the Fatou theorem.*

Let $f(z)$ be analytic and bounded in $|z|<1$ and let $F(z)$ be an indefinite integral of $f(z)$. Then $F(z)$ is continuous on the closed circle $|z|\leqq 1$ and defines a boundary function $F(t)$, $t=e^{i\theta}$. The proof of the Fatou theorem consists of showing that $f(z)$ has a radial limit equal to the derivative $F'(t)$ wherever $F'(t)$ exists. In this note the author proves conversely that wherever the radial limit $f(e^{i\theta})=\lim_{r\to 1}f(re^{i\theta})$ exists the derivative $F'(t)=F'(e^{i\theta})$ exists and has the same value. Thus the boundary function $f(t)$ is identically equal to the derivative of its indefinite integral, in the sense that the two functions exist at precisely the same points and have there the same values. An example is constructed to show that the hypothesis of boundedness cannot be materially weakened. (Received April 1, 1941.)

234. L. H. Loomis: *A simple proof of the Fatou theorem.*

The Fatou theorem in its simplest formulation can be stated as follows: If $f(x)$ is analytic and bounded in the unit circle $|z|<1$ then the radial limit $\lim_{r\to 1} f(re^{i\theta})$ exists for almost all θ. The standard proof of this theorem is based on the fact that a function of a real variable having a Lipschitz constant has a derivative almost everywhere. The manipulations involved center around the Poisson integral formula. The present proof assumes the same basis but attains essential manipulatory simplification by using the Cauchy integral formula instead of the Poisson integral and substituting the half-plane for the circle. (Received April 1, 1941.)

235. E. R. Lorch: *The spectrum of linear transformations.*

Let T be a bounded linear transformation in a complex Banach space, and let C be a simple closed rectifiable curve lying entirely within the resolvent set R of T. Then the integral $(1/2\pi i)\int_C d\zeta/(\zeta I-T)$ exists and represents a projection P reducing T. $P=0$ if and only if every point interior to C lies in R. $P=I$ if and only if every point exterior to C lies in R. If C' represents another curve in R and P' its associated projection, then $PP'=P$ if C lies in the interior of C'; $PP'=0$ if C and C' lie exterior to each other. This leads to the construction of a homomorphism between an algebra of sets on the one hand and an algebra of projections reducing T on the other—this mapping giving the spatial manifold counterparts of a fundamental class of spectral sets. (Received February 13, 1941.)

236. K. L. Nielsen: *Concerning boundary value problems for linear differential equations when the boundary conditions are given by Stieltjes integrals.*

This paper deals with the boundary values problem formulated by $L(x,\lambda;y(x,\lambda))=f(x)$ and the boundary conditions $M_i(y)\equiv\sum_{k=1}^{n}\int_c^d y^{(k-1)}(t,\lambda)d\alpha_{i,k}(t)=0$, $(i=1,\cdots,n)$, where $L(x,\lambda;y(x,\lambda))$ is the differential polynomial $\sum_{k=0}^{n}\lambda^{H(n-k)}a_{n-k}(x,\lambda)y^{[k]}$, the operators M_i are linearly independent, the involved integrals are in the sense of Stieltjes and the $\alpha_{i,k}(x)$ are functions of bounded variation. Tamarkin (Mathematische Zeitschrift, vol. 7 (1927), pp. 1–54) presented developments under the assumption that the roots of the characteristic equation are distinct and that $H=1$. Trjitzinsky (Acta Mathematica, vol. 67 (1936), pp. 1–50) has obtained an asymptotic representation of the solution of the differential equation for x in the real interval (c,d) and the parameter λ in certain regions extending to infinity for the case where the roots of the characteristic equation are not distinct and H is allowed to exceed unity. The author takes Trjitzinsky's results and obtains the restrictions which are necessary on the $\alpha_{i,k}(x)$ and states a theorem determining the values of λ for which the nonhomogeneous boundary value problem is possible when (c, d) is the interval for which the Trjitzinsky existence theorem holds. (Received March 31, 1941.)

237. C. D. Olds: Expansion of an arbitrary function in terms of certain polynomials.

Let $\{P_i(z)\}$ be a set of polynomials which satisfies an equation of the type: $Q(z)/n! =\sum_{i=0}^{n}\alpha_i P_i(z)$. (When $\{P_i(z)\}$ are the Bernoulli polynomials, for example, $Q(z)=z^n$.) This property is used to show that under certain conditions an arbitrary function $f(z)$, regular in the unit circle, can be expanded in the form $f(z)=\sum_{i=0}^{\infty}\beta_i P_i(z)$. Moreover, for certain of these polynomials the expansion is shown to be valid by summing the series directly. In the case of the Bernoulli polynomials use is made of certain results concerning their maximum values due to D. H. Lehmer (American Mathematical Monthly, vol. 47 (1940), pp. 533–538). (Received March 8, 1941.)

238. G. H. Peebles: *On the behavior of series of polynomials orthogonal with respect to a weight function of changing sign.*

Most of the properties of polynomials orthogonal with respect to a non-negative weight function, which are needed to show convergence of the formal expansion of a function, are lost, if the weight function changes sign in the interval of orthogonality. The partial sums corresponding to a weight function $\rho(x)$ which changes sign a finite number of times are related very simply, however, to the partial sums corresponding to $\pi(x)\rho(x)$, where $\pi(x)$ is a polynomial or reciprocal of a polynomial such that $\pi(x)\rho(x)$ is non-negative on the interval of orthogonality. This relation is used to study the behavior of the partial sums of the expansions of functions in terms of polynomials orthogonal with respect to a weight function of changing sign. (Received March 14, 1941.)

239. Tibor Radó: *On convergence in length and convergence in area.*

The purpose of the paper is to complete and to extend various results in the literature on convergence in length and convergence in area. The specific tool is the following inequality. On a bounded measurable set E of a euclidean space, let there be given two three-dimensional vector-functions x_1, x_2. Then $[\int\|x_1\times x_2\|]^2\leqq[\int\|x_1\|\cdot\|x_2\|]^2-[\int x_1\cdot x_2]^2$, where the integrals are taken over E, provided only that the integrals involved exist in the Lebesgue sense. This inequality, which is an immediate consequence of the identity $(x_1\times x_2)^2=x_1^2x_2^2-(x_1\cdot x_2)^2$, is a generalization of the inequality (7) in Adams and Lewy, *On convergence in length*, Duke Mathematical Journal, vol. 1 (1935), pp. 19–26. (Received March 28, 1941.)

240. W. C. Randels: *On Bessel's inequality in abstract spaces.*

Bochner (Fundamenta Mathematicae, vol. 20, p. 262) has defined Fourier series for functions whose values are in abstract spaces, and has pointed out that Bessel's inequality need not hold. In this paper examples of spaces in which the inequality does hold are given and Bochner's counterexample is extended. (Received March 3, 1941.)

241. C. E. Rickart: *Integration in a convex linear topological space.*

A theory of integration is developed for multi-valued functions $F(\sigma)$ defined over a σ-field $\mathfrak{M}$ of subsets of an abstract set M and with values in a convex linear topological space $\mathfrak{X}$. The value of the integral is a closed set in $\mathfrak{X}$. The fundamental properties obtained by Kolmogoroff (Mathematische Annalen, vol. 103 (1930), pp. 654–696) for $\mathfrak{X}$ the space of real numbers are shown to carry over to the more general situation. The method of definition of the integral is an extension of that used by R. S. Phillips (Transactions of this Society, vol. 47 (1940), pp. 114–145). A notion of convergence for subsets of $\mathfrak{X}$ similar to the Hausdorff convergence for subsets of a metric space plays a central role throughout. If the integral is restricted to be single-valued, then a very general convergence theorem involving a notion of approximate convergence is obtained. The integral contains an integral of G. B. Price (Transactions of this Society, vol. 47, pp. 1–50) as well as the Kolmogoroff and Phillips integrals. (Received March 14, 1941.)

242. Raphaël Salem: *On some properties of symmetrical perfect sets.*

I. Let P be a symmetrical perfect set of measure zero constructed in $(0, 2\pi)$ by the dyadic process; E_p the set of 2^p equal intervals obtained after p operations; F_p the continuous non-decreasing function $(F_p(0)=0,\ F_p(2\pi)=1)$ equal to $k/2^p$ in the kth interval contiguous to E_p, and linear in each interval of E_p; F the limit of F_p. If c_n is the Fourier-Stieltjes coefficient of dF with respect to e^{nix} and $E(p)$ the measure of E_p, the series $\sum E^2[a \log n] \cdot |c_n|^2$ converges for $a > 3/2 \log 2$, its sum depending only on a. Thence follow some properties of sets of absolute convergence for trigonometrical series. II. The following theorem is proved: if N is any set of absolute convergence for trigonometrical series, the sum of N and of any finite set is also a set of absolute convergence. (Received March 6, 1941.)

243. Raphaël Salem: *On trigonometrical series whose coefficients do not tend to zero.*

If $\sum \rho_n \cos(nx-\alpha_n)$, with $\rho_n \geqq 0$, $\limsup \rho_n > \alpha > 0$, converges in a set E, and if the point $x=0$ belongs to E, then there is a sequence of integers $\{n_k\}$ such that $\sin n_k x$ tends to zero for every x belonging to E. If E is perfect and if F is any non-decreasing function constant in every interval contiguous to E but increasing from one interval to another, the Fourier-Stieltjes cosine-coefficients of rank $2n_k$ of dF tend to the greatest possible limit $(F(2\pi)-F(0))/\pi$ as $k\rightarrow\infty$. If E is a perfect set such that this condition is satisfied, then there exists a trigonometrical series whose coefficients do not tend to zero and which converges "almost everywhere" in the set E. (Received March 6, 1941.)

244. Henry Scheffé: *Linear differential equations with two-term recurrence formulas.*

Linear differential equations with two-term recurrence formulas (defined below) occur very often in applied mathematics. The equation $\sum_{j=0}^{n} p_j(x)y^{(j)}=0$ is assumed to have coefficients analytic at some $x=x_0$. Following Frobenius, a formal substitution $y=\sum_{\nu=0}^{\infty} c_\nu \xi^{\nu+r}$ is made where $\xi \equiv x-x_0$ if $x_0 \neq \infty$, $\xi \equiv 1/x$ if $x_0=\infty$. The formula obtained by equating to zero the coefficient of $\xi^{\mu+r}$ for integral values of μ is called the recurrence formula relative to the point x_0. For the existence of a two-term recurrence formula, that is, one of the type $f(\nu, r)c_\nu+g(\nu, r)c_{\nu-h}=0$, for $\nu \geqq h$, a positive integer, necessary and sufficient conditions on the coefficients $p_j(x)$ are obtained for some neighborhood of x_0. It is then further assumed that the $p_j(x)$ are given elsewhere by their analytic continuations. Transformations $z=a\xi^b$, $w=\xi^c y$ are exhibited which carry the differential equation into a generalized hypergeometric equation with indices $p \leqq q+1$. Interesting corollaries follow. (Received March 12, 1941.)

245. I. J. Schoenberg: *On absolutely convex functions.*

1. A function $f(x)$ is said to be *completely monotone* for $x>0$ if (1) $(-1)^n f^{(n)}(x) \geqq 0$ $(x>0;\ n=0, 1, 2, \cdots)$. The following theorem shows that these requirements are redundant. Theorem: $f(x)$ is completely monotone for $x>0$ if it satisfies, for $x>0$, the following conditions: (2) $f(x) \geqq 0$, $f'(x) \leqq 0$, $f''(x) \geqq 0$, and (3) $(-1)^n f^{(n)}(x) \geqq 0$ for infinitely many values of n (no matter how distantly spaced). Moreover, the third condition (2) may be replaced by the following weaker one: $\lim \inf_{x \to +\infty} f'(x) > -\infty$. 2. A function $f(x)$ is said to be *absolutely convex* for $-1<x<1$ if it satisfies in that range the inequalities $f^{(2n)}(x) \geqq 0$ $(n=0, 1, 2, \cdots)$. S. Bernstein has shown that $f(x)$ is analytic and regular in the domain bounded by the circle $|x| = 1/4$ and the four tangents to it through the points $x=1$, $x=-1$. It is shown here that $f(x)$ is analytic within the unit circle $|x| <1$, which is the best result possible. (Received February 7, 1941.)

246. H. M. Schwartz: *Sequences of Darboux-Stieltjes integrals.*

The results obtained for sequences of Stieltjes integrals (abstract 47-1-43) are extended to sequences of upper and lower Darboux-Stieltjes integrals taken in the sense of Lebesgue (*Leçons sur l'Intégration*, 2d edition). (Received March 1, 1941.)

247. W. R. Scott: *A lemma on the Weierstrass E-function.* Preliminary report.

Let there be given an integrand $f(x^1, x^2, x^3, X^1, X^2, X^3)$ which we abbreviate $f(x, X)$. We assume that f satisfies the usual differentiability and homogeneity conditions. Let there be given further a set of values x_0, X_0 such that (1) $f(x_0, X_0)>0$, (2) $f(x_0, X) \geqq 0$ and $E(x_0, X_0, X) \geqq 0$ for every vector X, where E is the Weierstrass E-function. Let Y_0 denote the vector with components $\partial f/\partial X^i$, $i=1, 2, 3$, taken at (x_0, X_0). Then there exists a $\delta=\delta(x_0, X_0)>0$ such that $f(x_0, X) \geqq \delta$ for every unit vector X such that the scalar product XY_0 is positive. This lemma is then applied to answer various questions concerning the lower semi-continuity of simple and double integrals suggested by the following papers; E. J. McShane, *Integrals over surfaces in parametric form*, Annals of Mathematics, (2), vol. 34 (1933); *Semi-continuity of integrals* . . . , Duke Mathematical Journal, vol. 2 (1936); T. Radó, *On the semi-continuity of double integrals in parametric form*, to appear in Transactions of this Society. (Received March 12, 1941.)

248. W. T. Scott and H. S. Wall: *Gronwall summability.*

It is shown that a necessary and sufficient condition for a Hausdorff mean $[H, c_n]$ to be a Gronwall mean (f, g) (T. H. Gronwall, *Summation of series and conformal mapping*, Annals of Mathematics, (2), vol. 33 (1932), pp. 101–117) is that $[H, c_n]$ be the product of (E, β) by (C, α), that is, $c_n=\beta^n/C_{n+\alpha,n}$, $(n=0, 1, 2, \cdots)$, where $0<\beta \leqq 1$, $\alpha>0$. The (f, g)-mean which is identical with this $[H, c_n]$-mean has $f(w)=\beta w/[1-(1-\beta)w]$, $g(w)=(1-w)^{-\alpha-1}$. It is also shown that a method of summation considered by W. A. Mersman (this Bulletin, vol. 44 (1938), pp. 667–673) is identical with (f, g) with $f(w)=[1-(1-w)^{1/2}]/[1+(1-w)^{1/2}]$, $g(w)=(1-w)^{-1}$. (Received February 25, 1941.)

249. J. A. Shohat: *On the best polynomial approximation for functions possessing derivatives.*

By virtue of the characteristic property of the polynomial of best approximation, of degree less than or equal to n, to $f(x)$ on a given interval, it may be also considered as a Lagrangean interpolation polynomial for $f(x)$. In this way are obtained very simply general theorems on best approximation in case $f^{(n+1)}(x)$ exists, also on the distribution of the points of deviation. (Received February 3, 1941.)

250. L. H. Swinford: *On Abel's equation.*

The substitution $y=\alpha+\beta z^{1/2}/\gamma+z^{1/2}$ allows one to replace Abel's equation $y'+g(x)y^2+f(x)y^3=0$ by a system of two first order equations, by which means new cases of integrability are found. (Received March 7, 1941.)

251. Otto Szász: *On convergence and summability of trigonometric series.*

A new type of necessary or sufficient condition has been given recently for the convergence of a Fourier series at a point. This combines a continuity property of the function with an order condition on the coefficients, and is generalized in this paper. Moreover, instead of a Fourier series, more general trigonometric series are considered, associating for example the termwise integrated series with a function. The results are closely connected with two summability methods introduced by Riemann and Lebesgue. (Received March 5, 1941.)

252. Alexander Weinstein: *On the decomposition of a Hilbert space by its harmonic subspace.*

It is shown that a lemma playing a central part in a recent paper of H. Weyl on the method of orthogonal projection in potential theory (Duke Mathematical Journal, vol. 7 (1940), pp. 411–444, Lemma 2) is an almost immediate consequence of a problem of the unified theory of eigenvalues of plates and membranes considered some years ago in a joint paper of N. Aronszajn and the author (Comptes Rendus de l'Académie des Sciences, Paris, vol. 204 (1937), p. 96). The proof based on these results does not require any special construction or computation. (Received February 7, 1941.)

Applied Mathematics

253. G. E. Hay: *The finite displacement of thin rods.*

The finite displacement of thin rods has been considered by G. Kirchhoff, who introduced approximation based on the thinness of the rod in a rather unsatisfactory manner. In the present paper the method of the tensor calculus is employed, and there is introduced a systematic method of approximation which involves the expansion of the fundamental equations as power series in a dimensionless parameter ϵ and permits a theoretical solution of the problem to any desired degree of accuracy. Finally, application of the theory is made to the problem of "straightening" certain thin rods by means of systems of forces applied to the ends. (Received March 13, 1941.)

254. A. E. Heins: *On the transformation theory of the solution of partial differential equations.* Preliminary report.

The regularity condition of the Fourier transform of a function defined over a