NOTE ON INTERPOLATION

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Let

$$(1)\qquad A_n:\ x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)}, \qquad -1 \leqq x_1^{(n)},\ 1 \geqq x_n^{(n)},$$

denote a set of n distinct points of the interval $-1 \leqq x \leqq +1$. If $f(x)$ is a given function defined in $-1 \leqq x \leqq +1$ we call the polynomial

$$(2)\qquad I_n(x; f) = \sum_{k=1}^{n} f(x_k^{(n)}) q_k^{(n)}(x)$$

an interpolation polynomial of $f(x)$ corresponding to the abscissas (1), where for the polynomials $q_1^{(n)}(x), q_2^{(n)}(x), \cdots, q_n^{(n)}(x)$

$$(3)\qquad q_k^{(n)}(x_k^{(n)}) = 1, \qquad q_k^{(n)}(x_i^{(n)}) = 0, \qquad i \neq k.$$

Then the polynomial (2) represents a polynomial which assumes the value $f(x_k^{(n)})$ at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$. The polynomials $q_k^{(n)}(x)$ $(k = 1, 2, \cdots, n)$ are called the fundamental polynomials of the interpolation corresponding to the set A_n. We consider the sequence $I_n(x; f)$ $(n = 1, 2, \cdots)$ under the condition of continuity of $f(x)$.

Let $\omega_n(x)$ be a polynomial of degree n, not identically zero, vanishing at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$ and let

$$(4)\qquad q_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})(x - x_k^{(n)})} \equiv l_k^{(n)}(x);$$

then (2) represents the nth Lagrange polynomial of $f(x)$ corresponding to the abscissas (1), which is the uniquely determined polynomial of degree $n-1$ which assumes the value $f(x_k^{(n)})$ at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$.

It is known that for a given arbitrary sequence $\{A_n\}$:

$$(5)\quad x_1^{(1)};\ x_1^{(2)}, x_2^{(2)};\ x_1^{(3)}, x_2^{(3)}, x_3^{(3)};\ \cdots;\ x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)};\ \cdots$$

there exists a continuous function $f(x)$ such that the sequence of the Lagrange interpolation polynomials is not uniformly convergent, even divergent at a preassigned point.[1]

[1] G. Faber, *Über die interpolarische Darstellung stetiger Funktionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 389–408. S. Bernstein, *Sur la limitation des valeurs d'une polynome $P_n(x)$ de degré n sur tout un segment par ses valeurs en $(n+1)$ points du segment*, Bulletin de l'Académie des Sciences de l'URSS, 1931, pp. 1025–1050. In the important special case $x_k^{(n)} = \cos(2k-1)\pi/2n$, that is, for the zeros $\omega_n(x) = \cos n(\arccos x)$ ($\omega_n(x)$ is the nth Tschebycheff polynomial) much more is known. I have proved the existence of a continuous function $f(x)$ for which

Let

$$v_k^{(n)}(x) = 1 - (x - x_k^{(n)}) \frac{\omega_n''(x_k^{(n)})}{\omega_n'(x_k^{(n)})}, \tag{6}$$

$$q_k^{(n)}(x) = v_k^{(n)}(x)\{l_k^{(n)}(x)\}^2; \tag{7}$$

then $I_n(x; f)$, the nth Hermite interpolation polynomial of $f(x)$, represents the uniquely determined polynomial of degree $2n-1$ for which $I_n(x_k^{(n)};f) = f(x_k^{(n)})$ $(k=1, 2, \cdots, n)$ and $I_n'(x_k^{(n)};f)=0$ $(k=1, \cdots, n)$. There are sequences $\{A_n\}$ for which the sequence of Hermite interpolation polynomials is uniformly convergent for an arbitrary continuous function.[2]

The question arises whether it is possible to determine, for a given arbitrary sequence $\{A_n\}$, the fundamental polynomials

$$\begin{gathered} q_1^{(1)}(x); \quad q_1^{(2)}(x), \ q_2^{(2)}(x); \quad q_1^{(3)}(x), \ q_2^{(3)}(x), \ q_3^{(3)}(x); \ \cdots; \\ q_1^{(n)}(x), \ q_2^{(n)}(x), \cdots, q_n^{(n)}(x); \ \cdots \end{gathered} \tag{8}$$

so that for an arbitrary continuous function $f(x)$

$$\lim_{n=\infty} I_n(x;f) = \lim_{n=\infty} \sum_{k=1}^{n} f(x_k^{(n)}) q_k^{(n)}(x) = f(x) \tag{9}$$

holds uniformly in $-1 \leqq x \leqq 1$.

For the sequence $\{A_n\}$ we must suppose that the set (5) is everywhere dense in $-1 \leqq x \leqq +1$, that is, $\max_{(i)} (x_{i+1}^{(n)} - x_i^{(n)}) \to 0$ when $n \to \infty$. We prove that this trivial necessary condition is however sufficient to construct the fundamental polynomials (8) so that (9) holds for every continuous $f(x)$.

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the sequence of Lagrange polynomials corresponding to these abscissas is everywhere divergent, even everywhere unbounded. G. Grünwald, *Über Divergenzerscheinungen der Lagrange'schen Interpolationspolynome stetiger Funktionen*, Annals of Mathematics, (2), vol. 37 (1936), pp. 908–918. See also J. Marcinkiewicz, *Sur la divergence des polynomes d'interpolation*, Acta Litterarum ac Scientiarum, Szeged, vol. 8 (1937), pp. 131–135.

[2] Such a sequence is, for example, $x_k^{(n)} = \cos (2k-1)\pi/2n$. See L. Fejér, *Über Weierstrass'sche Approximation, besonders durch Hermite'sche Interpolation*, Mathematische Annalen, vol. 102 (1930), pp. 707–725. It was Fejér who began the investigation of the Hermite interpolation polynomials instead of and beside the Lagrange polynomials, and the fundamental methods and theorems of the theory are due to him. See, for example, L. Fejér, *On the characterisation of some remarkable systems of points of interpolation by means of conjugate points*, American Mathematical Monthly, vol. 41, (1939), pp. 1–14.

Let $\phi_k^{(n)}(x)$ be the continuous function defined in the following way: $\phi_1^{(1)}(x)\equiv 1$; for $n>1$, and $k\neq 1, n$,

$$\phi_k^{(n)}(x) = \begin{cases} 0 \text{ in } -1 \leqq x \leqq x_{k-1}^{(n)}, \\ \text{linear in } x_{k-1}^{(n)} \leqq x \leqq \frac{1}{2}(x_{k-1}^{(n)} + x_k^{(n)}), \\ 1 \text{ in } \frac{1}{2}(x_{k-1}^{(n)} + x_k^{(n)}) \leqq x \leqq x_k^{(n)}, \\ \text{linear in } x_k^{(n)} \leqq x \leqq \frac{1}{2}(x_k^{(n)} + x_{k+1}^{(n)}), \\ 0 \text{ in } \frac{1}{2}(x_k^{(n)} + x_{k+1}^{(n)}) \leqq x \leqq +1; \end{cases}$$

for $k=1, n$

$$\phi_1^{(n)}(x) = \begin{cases} 1 \text{ in } -1 \leqq x \leqq x_1^{(n)}, \\ \text{linear in } x_1^{(n)} \leqq x \leqq \frac{1}{2}(x_1^{(n)} + x_2^{(n)}), \\ 0 \text{ in } \frac{1}{2}(x_1^{(n)} + x_2^{(n)}) \leqq x \leqq +1; \end{cases}$$

$$\phi_n^{(n)}(x) = \begin{cases} 0 \text{ in } -1 \leqq x \leqq x_{n-1}^{(n)}, \\ \text{linear in } x_{n-1}^{(n)} \leqq x \leqq \frac{1}{2}(x_{n-1}^{(n)} + x_n^{(n)}), \\ 1 \text{ in } \frac{1}{2}(x_{n-1}^{(n)} + x_n^{(n)}) \leqq x \leqq +1. \end{cases}$$

We have evidently

$$\sum_{k=1}^{n} \phi_k^{(n)}(x) \equiv 1. \tag{10}$$

Let $q_k^{(n)}(x)$ be a polynomial for which

$$q_k^{(n)}(x_i^{(n)}) = \phi_k^{(n)}(x_i^{(n)}), \qquad i = 1, 2, \cdots, n, \tag{11}$$

$$\left| q_k^{(n)}(x) - \phi_k^{(n)}(x) \right| < 1/n^2, \qquad -1 \leqq x \leqq +1. \tag{12}$$

The existence of such a polynomial follows from a slight modification of the Weierstrass approximation theorem.[3] Equation (3) follows from (11), and from (12) and (10) we have

$$\begin{aligned} \left| \sum_{k=1}^{n} q_k^{(n)}(x) - 1 \right| &= \left| \sum_{k=1}^{n} (q_k^{(n)}(x) - \phi_k^{(n)}(x)) \right| \\ &\leqq \sum_{k=1}^{n} \left| q_k^{(n)}(x) - \phi_k^{(n)}(x) \right| \leqq \frac{1}{n}\cdot \end{aligned} \tag{13}$$

[3] This modification is as follows: Let $f(x)$ be continuous in $-1\leqq x\leqq +1$ and $x_1, x_2, \cdots, x_n$ given numbers in $-1\leqq x\leqq +1$. Then for any $\epsilon>0$ there exists a polynomial $P(x)$ so that $|P(x)-f(x)|<\epsilon$ and $P(x_i)=f(x_i)$, $i=1, 2, \cdots, n$.

Let x be a fixed number in the interval $-1 \leqq x \leqq +1$ and $\epsilon > 0$. Because of the continuity of the function $f(x)$ if n is large enough and $|x - x_k^{(n)}| < \delta(\epsilon) = \delta$ we have

$$|f(x) - f(x_k^{(n)})| < \epsilon/2;$$

such $x_k^{(n)}$ exist if n is large enough because the sequence is everywhere dense. Then we have from (13)

$$\begin{aligned}
|I_n(x;f) - f(x)| &= \left| \sum_{k=1}^{n} f(x_k^{(n)}) q_k^{(n)}(x) - f(x) \right| \\
&= \left| \sum_{k=1}^{n} (f(x_k^{(n)}) - f(x)) q_k^{(n)}(x) - f(x) + f(x) \sum_{k=1}^{n} q_k^{(n)}(x) \right| \\
&\leqq \sum_{k=1}^{n} |f(x) - f(x_k^{(n)})| \, |q_k^{(n)}(x)| \\
&\quad + |f(x)| \left| \sum_{k=1}^{n} q_k^{(n)}(x) - 1 \right| \\
&\leqq \sum_{|x - x_k^{(n)}| > \delta} |f(x) - f(x_k^{(n)})| \, |q_k^{(n)}(x)| \\
&\quad + \frac{\epsilon}{2} \sum_{|x - x_k^{(n)}| < \delta} |q_k^{(n)}(x)| + |f(x)|/n \\
&\leqq 2M \sum_{|x - x_k| > \delta} |q_k^{(n)}(x)| + \frac{\epsilon}{2}(1 + 1/n) + M/n,
\end{aligned} \tag{14}$$

where $M = \max_{-1 \leqq x \leqq +1} |f(x)|$. For large n and fixed δ, $\phi_k^{(n)}(x) = 0$ if $|x - x_k^{(n)}| > \delta$, so it follows from (12) that $|q_k^{(n)}(x)| < 1/n^2$ if $|x - x_k^{(n)}| > \delta$, (14) gives for large n

$$|I_n(x;f) - f(x)| < 2M/n + \tfrac{1}{2}\epsilon(1 + 1/n) + M/n < \epsilon,$$

which was to be proved.

BUDAPEST, HUNGARY