SPACE CREMONA TRANSFORMATIONS OF ORDER $m+n-1$[1]

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1. **Introduction.** This paper discusses a space Cremona transformation of order $m+n-1$ (m, n any integers) generated by two rational twisted curves. One special position of the defining curves gives rise to an involution recently described,[2] while another special position results in an involution somewhat similar to one which was defined in a different manner by Montesano.[3]

2. **Cremona transformation.** Consider a curve C_n of order n having $n-1$ points on each of two skew lines d and d', and a curve C_m' of order m having $m-1$ points on each of d and d' (m, n, any integers). A generic point P determines a ray through it intersecting C_n once in α and d once in β. P also determines a ray through it intersecting C_m' once in γ and d once in δ. We define P', the correspondent of P, to be the intersection of lines $\alpha\delta$ and $\beta\gamma$.

It is to be noted that if C_n should become identical with C_m' but d and d' remain distinct, there would result the Cremona involution we discussed in a recent paper (loc. cit.).

Let the equations of d be $x_1=0$, $x_2=0$, and those of d' be $x_3=0$, $x_4=0$. Let C_n be

$$x_1 = (as + bt) \prod_1^{n-1} (t_is - s_it), \qquad x_2 = (cs + dt) \prod_1^{n-1} (t_is - s_it),$$

$$x_3 = (es + ft) \prod_n^{2n-2} (t_is - s_it), \qquad x_4 = (gs + ht) \prod_n^{2n-2} (t_is - s_it),$$

where s_i, t_i for $i=1, 2, \cdots, n-1$ are values of the parameters of C_n for points on d, and for $i=n, n+1, \cdots, 2n-2$, for points on d'.

Let the equations of C_m' be

$$x_1 = (AS + BT) \prod_1^{m-1} (T_iS - S_iT), \quad x_2 = (CS + DT) \prod_1^{m-1} (T_iS - S_iT),$$

$$x_3 = (ES + FT) \prod_m^{2m-2} (T_iS - S_iT), \quad x_4 = (GS + HT) \prod_m^{2m-2} (T_iS - S_iT),$$

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[1] Presented to the Society, September 10, 1940.

[2] E. J. Purcell, *A multiple null-correspondence and a space Cremona involution of order* $2n-1$, this Bulletin, vol. 46 (1940), pp. 339–444.

[3] D. Montesano, *Su una classe di trasformazioni involutorie dello spazio*, Rendiconti del' Istituto Lombardo di Scienze e Lettere, (2), vol. 21 (1888), pp. 688–690.

where S_i, T_i for $i=1, 2, \cdots, m-1$ are values of the parameters of C_m' for points on d, and for $i=m, m+1, \cdots, 2m-2$, for points on d'. Then the equations of the transformation are

$$x_1' = k(Q_1x_3 + Q_2x_4)\left(\prod_1^{n-1} \theta_i\right)\left(\prod_1^{m-1} \Phi_i\right),$$

$$x_2' = k(R_1x_3 + R_2x_4)\left(\prod_1^{n-1} \theta_i\right)\left(\prod_1^{m-1} \Phi_i\right),$$

$$x_3' = K'(r_2x_1 - q_2x_2)\left(\prod_n^{2n-2} \theta_i\right)\left(\prod_m^{2m-2} \Phi_i\right),$$

$$x_4' = K'(q_1x_2 - r_1x_1)\left(\prod_n^{2n-2} \theta_i\right)\left(\prod_m^{2m-2} \Phi_i\right),$$

where $k\equiv(bc-ad)$, $K'\equiv(FG-EH)$, and

$$\begin{aligned}
Q_1 &\equiv (AH - BG), & Q_2 &\equiv (BE - AF),\\
R_1 &\equiv (CH - DG), & R_2 &\equiv (DE - CF),\\
q_1 &\equiv (ah - bg), & q_2 &\equiv (be - af),\\
r_1 &\equiv (ch - dg), & r_2 &\equiv (de - cf),
\end{aligned}$$

$$\theta_i \equiv \{t_i(bx_2 - dx_1) - s_i(cx_1 - ax_2)\},$$

$$\Phi_i \equiv \{T_i(Hx_3 - Fx_4) - S_i(Ex_4 - Gx_3)\}.$$

The inverse transformation is

$$x_1 = K(q_1x_3' + q_2x_4')\left(\prod_1^{n-1} \phi_i'\right)\left(\prod_1^{m-1}\Theta_i'\right),$$

$$x_2 = K(r_1x_3' + r_2x_4')\left(\prod_1^{n-1} \phi_i'\right)\left(\prod_1^{m-1}\Theta_i'\right),$$

$$x_3 = k'(R_2x_1' - Q_2x_2')\left(\prod_n^{2n-2} \phi_i'\right)\left(\prod_m^{2m-2}\Theta_i'\right),$$

$$x_4 = k'(Q_1x_2' - R_1x_1')\left(\prod_n^{2n-2} \phi_i'\right)\left(\prod_m^{2m-2}\Theta_i'\right),$$

where $K\equiv(BC-AD)$, $k'\equiv(fg-eh)$,

$$\phi_i' \equiv \{t_i(hx_3' - fx_4') - s_i(ex_4' - gx_3')\},$$

$$\Theta_i' \equiv \{T_i(Dx_1' - Bx_2') - S_i(Ax_2' - Cx_1')\}.$$

Both the direct and inverse transformations are of order $m+n-1$, where m and n are any integers.

The fundamental system and its images for the direct transformation are as follows.

d is an $(n-1)$-fold F-line of simple contact. The fixed tangent planes are $\theta_i=0$, where $i=1, 2, \cdots, n-1$. It is of the first species and its P-surface consists in the planes $\phi_i'=0$, where $i=1, 2, \cdots, n-1$, which pass through d'.

d' is an $(m-1)$-fold F-line of simple contact. The fixed tangent planes are $\Phi_i=0$, where $i=m, m+1, \cdots, 2m-2$. It is of the first species and its P-surface consists in the $m-1$ planes $\Theta_i'=0$ through d, where $i=m, m+1, \cdots, 2m-2$.

Each of the $m-1$ intersections of C_m' and d is an n-fold isolated F-point. Their P-surfaces are $\Theta_i'=0$, where $i=1, 2, \cdots, m-1$, respectively.

Each of the $n-1$ intersections of C_n and d' is an m-fold isolated F-point. Their P-surfaces are $\phi_i'=0$ $(i=n, n+1, \cdots, 2n-2)$ respectively.

The $(n-1)(m-1)$ lines of intersection of the $n-1$ fixed tangent planes through d with the $m-1$ fixed tangent planes through d' are simple F-lines without contact. They are of the second species.

The $(m-1)(n-1)$ lines joining the $m-1$ n-fold isolated F-points on d with the $n-1$ m-fold isolated F-points on d' are simple F-lines without contact. They are of the second species.

We may obtain a description of the fundamental system of the inverse transformation by interchanging m and n, C_n and C_m', θ_i and Θ_i', Φ_i and ϕ_i', wherever they appear in the foregoing.

C_n, d, and d' lie on the same quadric surface Q, and C_m', d, and d' lie on a quadric surface Q'. These quadrics may be the same or distinct and, while this does not affect the preceding discussion, the invariant systems for the two cases are different.

When Q and Q' are distinct, they intersect in d, d', and two transversals l_1 and l_2. The d and d' are common generators of the μ-systems of the two quadrics, while l_1 and l_2 are common generators of their λ-systems. The transformation sends each λ-generator of Q over into a λ-generator of Q', and each λ-generator of Q' over into a λ-generator of Q. Thus Q as a whole corresponds to Q' and vice versa. Each λ-generator of either quadric belongs to a cycle of index four—that is, four applications of the transformation leave every λ-generator invariant. The transformation interchanges C_n and C_m'. The points of l_1 are in involution; thus l_1 is an invariant line and the two fixed points of the involution are invariant points. Similarly for l_2. These four invariant points are the only invariant points that are not also F-points.

Let us now consider the case where C_n, C_m', d, and d' all lie on the same quadric $Q \equiv x_1x_4 - x_2x_3 = 0$. The transformation causes C_n and C_m' to interchange. The pencil of planes $x_4 - \lambda x_3 = 0$ is in involution with the pencil $x_2 - \lambda x_1 = 0$ and this makes each λ-generator of Q invariant. Consequently Q is invariant. The locus of invariant points is a curve K_{m+n} of order $m+n$ lying on Q. K_{m+n} passes through the $m+n-2$ points of intersection of C_n and C_m' and intersects d and d' in the $m+n-2$ isolated F-points on each of them. It intersects every μ-generator of Q in $m+n-2$ points and intersects every λ-generator in two points.

3. **Involution.** Consider a twisted curve C_n having $n-1$ points Δ_i on a straight line d, and a curve C_m' having $m-1$ points Σ_i on the same straight line d (m, n any integers). A generic point P determines a ray through it intersecting C_n in α and d in β, and also a ray through it intersecting C_m' in γ and d in δ. We define P', the correspondent of P in the involution, to be the intersection of lines $\alpha\delta$ and $\beta\gamma$.

If, in §2, we make d and d' identical, we obtain an involution of this kind. However, the curves C_n and C_m' of the present section do not necessarily lie on quadric surfaces.

The involution is of order $m+n-1$.

The fundamental system and its principal images follow.

d is an $(n+m-2)$-fold F-line of simple contact. The fixed tangent planes are $\theta_i = 0$, where $i = 1, 2, \cdots, n-1$, and $\Theta_i = 0$, where $i = 1, 2, \cdots, m-1$. It is of the second species and counts $(n+m-1) \cdot (n+m-2)$ times in the intersection of any two homaloids.

Points Δ_i are isolated F-points. Their P-surfaces are the planes $\theta_i = 0$ $(i = 1, 2, \cdots, n-1)$ respectively.

Points Σ_i are isolated F-points. The P-element of each is $\Theta_i = 0$ $(i = 1, 2, \cdots, m-1)$ respectively.

As we have seen, a general point P determines with d a plane π intersecting C_n in α and C_m' in γ. Call L the intersection of lines $\alpha\gamma$ and d. Then J, the harmonic conjugate of L with respect to α and γ, will be the only invariant point of π other than points of d. As π makes one revolution about d, α moves in π and crosses d $n-1$ times; γ also moves in π, crossing d $m-1$ times. As α approaches d, J approaches the same point on d, and the locus of J intersects d in all the points d has in common with C_n and C_m'. The locus of J is a rational curve K_{m+n-1} of order $m+n-1$ having $m+n-2$ points on d. K_{m+n-1} is the locus of invariant points.

It is clear that the line PP' intersects K_{m+n-1} in J and d in L, and that P and P' are harmonic conjugates with respect to[4] J and L.

[4] Compare with Montesano, loc. cit.

4. **Lower order for particular positions of the defining elements.** Each of the fixed tangent planes $\theta_i = 0$ mentioned in the contact conditions for the involution passes through d and is tangent to C_n at the corresponding Δ_i. The fixed tangent planes $\Theta_i = 0$ are similarly related to the curve C_m'.

If C_n and C_m' are so situated that a plane of $\theta_i = 0$ $(i = 1, 2, \cdots, n-1)$ coincides with a plane of $\Theta_i = 0$ $(i = 1, 2, \cdots, m-1)$, the order of the involution is reduced by one. In this way we may reduce the order by any integer up to, and including, the smaller of the two numbers $n-1$ and $m-1$.

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