the product for a second multiplication was used instead of factorial and logarithm approximations, and that all probabilities were computed in at least two different ways, leads the authors to make highly confident statements as to the accuracy of the figures.

The reviewer's reaction to a first reading of this work was mainly a sense of increased appreciation of the fundamentals and potentialities of the game—much as one gets from a study of a good chess book. Further study of some of the vital points would undoubtedly save many a trick and generally improve the game of most of us.

JOSEPH A. GREENWOOD

Introduzione al Pensiero Matematico. By Friedrich Waismann. Translated by Ludovico Geymonat. (Biblioteca di Cultura Scientifica, vol. 111.) Torino, Einaudi, 1939. 325 pp. Lire 20.

This book, originally published in German (Vienna, 1936) and now translated into Italian, is a popular exposition of the fundamental concepts and views underlying modern mathematics. Special attention is paid to those sides of present day mathematics which may seem to involve "philosophical" problems, and a manifold of philosophical suggestions are advanced, mostly credited to an unpublished manuscript by Wittgenstein. The expository part of the book combines, to a rare degree, accuracy and comprehensibility.

After having traced in outline the historical growth of our present numbers system, the author envisages the question, often raised by philosophers: "How can the postulation of 'new' numbers be justified? Do there 'exist,' e.g., any irrational and complex numbers?" It is shown that these questions are equivalent to the questions whether the enlarged mathematical calculus is consistent, and whether it can be given an interpretation. The answer by a reference to geometrical facts is rejected as unsatisfactory, since proof of the consistency of geometry will depend on the assumption that our number system is consistent. The problem of the existence of the non-natural numbers—as far as it can be separated from the consistency problem—is now solved by giving the well known construction of them on the basis of the natural numbers. Following Skolem, the author then shows how elementary arithmetic can be strictly developed on the minimum basis of three undefined notions (natural number, successor, identity) and one inductive definition (of addition) by using the method of complete induction. The present situation of the "Grundlagenforschung" is reviewed, mention being made of theorems by Gödel and Skolem.

The "philosophical" ideas, expressed in the book, are unfortunately presented in such a condensed form that it is difficult to judge their significance. I will here make a few critical comments, which should in no way obscure the high value of the book as an elementary exposition of the foundations of modern mathematics.

Throughout, the author stresses the important distinction between mathematics as a symbolic calculus and the various possible interpretations of this calculus. A strict observation of this distinction, however, should have led to more careful statements at a few points. When talking "philosophically" about what, for example, the natural numbers *are*, our statements must always be referred to some particular interpretation of arithmetic. But, in Chapters IV and V, it is said unconditionally that natural numbers, integers, and rational numbers *are* of three different logical types. This contention is, of course, true if we presuppose—as is done in the present book—the modern construction of integers and rational numbers. No reasons, however, have been given why there should not exist interpretations of another kind, and perhaps one which at once would give the natural numbers their usual meaning and make natural numbers, integers, and rational numbers all belong to the same logical type.

In Chapter VIII, the method of mathematical induction is discussed and a philosophical interpretation, due to Wittgenstein, is presented. This interpretation may be summarized thus. The sentence "Every (natural number) a has (the property) P," means, "1 has P, and if b has P, then $b+1$ has P." Following Skolem the author has not made use of the operators "for every x" and "for some x" in his development of elementary arithmetic. What he wants to interpret must, therefore, be the meaning of a statement, "a has P," where generality is expressed by the free variable a. The circularity of his explanation is then obvious. The positivistic principle of verification is invoked as a substantiation. This principle, as applied by the author to mathematics, could, I think, be formulated somewhat like this: "The meaning of a theorem p is identical with (the meaning of) the conjunction of those theorems which constitute the proof of p." This principle is open to a number of more or less obvious objections. If the words within parentheses in the above formulation are omitted, then it boils down to a purely syntactical definition which does not have anything to do with "meaning" in an interpretative sense. Whether the author would let them stand or not, cannot, however, be gathered from his presentation. But further: The principle leads to the consequence that only provable sentences have meaning, a consequence which seems absurd if "meaning" is taken in the interpretative sense. And what is "*the* proof" of a mathematical theorem? From the principle of verification, by the way, several other conclusions are drawn. One of them is the Brouwerian contention that the negation of a universal sentence is not equivalent to an existential sentence. This conclusion is clearly possible only if we are already arguing on the basis of an axiomatization of mathematics in accordance with Brouwer's view!

In Chapter IX: B the Frege-Russell interpretation of natural numbers is criticized. The discussion is centered around their well known definition of the phrase "the number of elements in the two sets A and B is the same" as meaning "there is a one-one relation of which A is the domain and B the converse domain" ("A"). The chief objections made against this definition are: (1) According to the definition, the number of elements in two sets A and B must either be the same or not be the same, whereas in reality there are sets A and B such that neither the sentence "A and B have the same number of elements" ("B") nor its verbal negation ("C") is true, if we take this sentence with its accepted meaning. A contention like this, of course, cannot be conclusively proved or refuted. But the alleged instance of two such sets, mentioned by the author (p. 156), does not seem convincing. The argument is essentially dependent on the obscure principle of verification, here applied to empirical sentences. However, even if the contention were true, it is still not clear in what way it would be an objection to the Frege-Russell analysis. The contention seems to allow two different interpretations: (i) By C, the verbal negation of B, we ordinarily understand the logical negation of B; but the law of the excluded middle is not universally valid. (ii) When B and C, as ordinarily used, are meaningful, then C is the logical negation of B; but in the case at hand—by force of the principle of verification—they are both deprived of meaning, and therefore the law of the excluded middle is not relevant. If (i) covers the author's intentions—why could not A, too, be a sentence for which the law of the excluded middle does not hold? If (ii) is meant—why could not the principle of verification sometimes make A and non-A meaningless, too? (2) Two sets, A and B might have the same number of elements—the phrase being taken in its usual meaning—in spite of the fact that no one-one relation held between them; the occurrence of such a relation is an accidental empirical fact. Here the term "relation" has obviously been taken in a restricted sense as synonymous with "physical relation," a sense not intended by the Frege-Russell theory. (The question has been discussed by Ramsey in his *Foundations of Mathematics*.)

In a concluding chapter (to be compared with Chapter IX: C), the philosophical standpoint of the book is restated. In opposition to the view that the rules governing the use of a sign follow from the meaning of the sign, it is asserted that the sign obtains its meaning just through the rules. The rules the author has in mind seem to be of the following kinds: (1) the syntactical rules of the mathematical calculus within which the sign occurs; (2) the rules for translating the mathematical formulae into our everyday language (e.g., "two plus two make four" is a translation of "$2+2=4$"); (3) the rules for integrating the mathematical calculus into everyday language (e.g., from "$2+2=4$" we may deduce "two apples plus two apples make four apples"). Given these rules, the meaning of "1," "2," and so on is determined. To that extent, the author's contention that "the rules give meaning to the signs" is obviously true. The words "one," "two," and so on are, in their everyday usage, endowed with a certain meaning, and by the rules, belonging to category (2), this meaning is transferred to "1," "2," and so on. But, on the other hand, if we let "1," "2," $\cdots$ mean what is ordinarily meant by "one," "two," $\cdots$, then we cannot arbitrarily change the rules of categories (1) and (3) and still obtain true mathematical formulae and valid deductions. So far, the view which the author rejects seems to be a commonplace. Possibly, the author would add to the above list of rules a fourth category: rules for deciding when empirical numerical sentences are true. Through these rules, he would say, words like "one," "two" get their meaning, and thus, ultimately, rules of usage constitute the meaning of mathematical terms. But—can we give any rule for the use of the term "two" that is not of the following type: "If and only if the kind A has the property P then the sentence 'There are two A's' is true"? If, as it seems to be, we cannot, then we have reduced the meaning of "two" to the meaning of "P" which latter meaning we have not explained in terms of any rules of usage.

It might be worthwhile mentioning that the theory of natural numbers, given in Chapter VIII, suffers from a somewhat subtle logistic fault. Letting his variables range over a domain of objects, called "natural numbers," the author takes the monadic successor function $a+1$ as his undefined concept besides the relation of identity. On this basis he gives the following "inductive" definition of the dyadic additive function: $a+(b+1)=(a+b)+1$. To be in the standard from, this definition should be completed by another equation: $a+1=\cdots$. The omission here suggests a confusion of the successor function $a+1$ (argument a) with the additive function $a+1$ (arguments a and 1).

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