A CHARACTERIZATION OF EUCLIDEAN SPACES

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The purpose of this paper is to give an elementary proof of the fact that a Banach space in which there exist projection transformations of norm one on every two-dimensional linear subspace is a euclidean space. S. Kakutani [1] has pointed out that a modification of a proof due to Blaschke [2] will prove this theorem. F. Bohnenblust has been able to establish this theorem for the complex case by still another method.[1]

A Banach space is a linear, normed, complete space [3, chap. 5]. A euclidean space of dimension α, where α is any cardinal number, is defined to be the Banach space of sequences x_ν of real numbers where ν ranges over a class of cardinal number α, and $\sum x_\nu^2$ is finite and equal to the square of the norm [4]. We consider only spaces having at least three linearly independent elements.

P. Jordan and J. von Neumann have shown [5] that a Banach space which is euclidean in every two-dimensional linear subspace is itself a euclidean space. It is thus sufficient to show that the "unit sphere" S for any three-dimensional linear subspace is an ellipsoid.

Because of the norm properties, S is a convex body symmetric about the origin o, and contains o as an interior point. Let γ be a plane containing o and let C_γ be the curve of intersection of γ and the boundary S' of S. The existence for each γ of a projection operation of norm one, whose direction of projection is that of the unit vector v_γ, implies that the cylinder generated by lines of direction v_γ tracing C_γ contains S. Our theorem is therefore an immediate consequence of the following lemma on convex bodies (which need not be symmetric about o).

LEMMA.[2] *If S is a convex body such that for every γ there exists a cylinder generated by C_γ containing S, then S is an ellipsoid.*

We topologize the planes γ by representing each by its direction cosines as a point on the unit sphere and using the usual topology of the unit sphere.

The proof of the lemma is divided into two parts. We first show that v_γ is uniquely determined by γ, that v_γ is a continuous function of γ, and that S' has a tangent plane at each of its points. It is then

[1] F. Bohnenblust's result is not yet published.

[2] W. Blaschke has proved a similar theorem under the assumption that there exists a tangent plane at each point of S' [2].

easy to demonstrate that the curves of intersection for any set of parallel planes are similar. Finally we prove that such a convex body is an ellipsoid.

We state without proof the following elementary propositions about boundaries of convex bodies. S' and C_γ are homeomorphs of the two-sphere and the one-sphere respectively. At each point of C_γ there exist two one sided tangents. The right (or left) sided tangents at points q approach the right (or left) sided tangents at p as q approaches p from the right (or left). Furthermore C_γ has a tangent at all except a denumerable set of its points. Given any one sided tangent t at p to C_γ, there exists a plane of support at p containing[3] t. Finally given a tangent t at p to C_γ there exist two half tangent planes to S' at p each containing t; that is, for any $\beta \neq \gamma$ containing the line $\overline{op}$, each one sided half tangent to C_β at p is contained in one of two half-planes, whose common bound is[4] t.

Suppose C_γ has a tangent t at p and that P_1, P_2 are the half tangent planes at p to S'. P_1 and P_2 may or may not be distinct. Let $d(x)$ be the distance from $x \in S'$ to the closest of P_1 and P_2, and let $r(x)$ be the distance from $x \in S'$ to p. The convexity of S then implies that $d(x) \leqq e(r) \cdot r(x)$ where $e(r) \rightarrow 0$ with r. For a $\beta \neq \gamma$ containing the line $\overline{op}$, we consider any $x \in C_\beta$ and denote by P_x the plane P_1 or P_2 from which $d(x)$ is measured. Then in order that the line through x in the direction of v_β does not pierce S', it is necessary that the component of v_β perpendicular to P_x be less than $2 \cdot e(2r(x))/\sin(\gamma, \beta)$ which approaches zero as $x \rightarrow p$. Hence $p + v_\beta$ is contained in both P_1 and P_2.

It is now easy to show that v_γ is uniquely determined by γ. Suppose two such directions of projection existed and let p be a point of tangency on C_γ with the tangent t. Then there exist two distinct half tangent planes containing t as their common line. The above argument shows that for any β containing $\overline{op}$, $p + v_\beta = t$. But this is impossible for a convex body with interior points.

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[3] Given a one-sided tangent t at p to C_γ, there exists a plane of support at p containing t. C_γ is convex. Therefore C_γ lies entirely on one side of any plane containing t. If the theorem were false, then we could find a plane containing t and interior points of S on both sides of the line through t. The convex extension of some neighborhoods about these points belongs to S and intersects γ on both sides of the line through t, which is impossible.

[4] Given a tangent t at p to C_γ, there exist two half tangent planes at S' at p each containing t. If the contrary were true, there would exist a β containing the line $\overline{op}$ such that a half tangent t' to C_β at p did not determine a plane of support with t at p. The plane through t, t' therefore contains an interior point of S. Again the convex extension of some neighborhood of such a point and C_γ belongs to S and intersects β on both sides of the line through t', which is impossible.

If $v_{\gamma_n} \to v_0$ as $\gamma_n \to \gamma$, then v_0 defines a direction of projection for γ. If the contrary were true, there would exist $p \in C_\gamma$ and a real number λ such that $q = p + \lambda v_0$ is an interior point of S. Let β be the plane determined by o, p, q and p_n the point of $C_{\gamma_n} \cdot C_\beta$ closest to p. Then, as $n \to \infty$, $p_n \to p$ and the points not interior to S, $p_n + \lambda v_{\gamma_n} \to q$, which is impossible. By the uniqueness, $v_0 = v_\gamma$. Since the v_γ form a compact set, it follows that v_γ is a continuous function of γ.

We next show that S' has a tangent plane at each of its points. Let $p \in S'$, γ contain $\overline{op}$, and let t be, say, the right sided tangent at p to C_γ. Then there exist points $p_n \in C_\gamma$ which approach p from the right and which have tangents t_n to C_γ. Let us choose a subsequence for which a set of half tangent planes $P_{n'}$ converge. For convenience we renumber this subsequence $1, 2, \cdots, n, \cdots$. Suppose $P_n \to P$. Then P contains t. Finally for any $\beta \neq \gamma$ containing $\overline{op}$, let us choose a sequence β_n containing $\overline{op_n}$ such that $\beta_n \to \beta$. As above, $p_n + v_{\beta_n}$ lies in P_n and since $v_{\beta_n} \to v_\beta$, $p + v_\beta$ lies in P. As v_β does not lie in β, it follows that the v_β for β containing $\overline{op}$ determine P. But t was an arbitrary one-sided tangent at p. Hence P contains all one sided tangents to curves C_β (β containing $\overline{op}$) and is therefore the tangent plane to S' at p.

We now define any directed line through o to be the z axis. The x-y plane is then the plane containing o which is parallel to the tangent plane to S' at the intersection p of S' and the z axis. In a system of cylindrical coordinates, let γ_θ be the plane $\theta = \text{const}$. Then v_{γ_θ} lies in the x-y plane. The curve of intersection C_z of S' with the plane $z = \text{const.}$ is defined by the differential equation

$$dr/d\theta = rF(\theta), \qquad r(0) = f(z)$$

where $F(\theta)$ is a continuous function independent of z. S' is therefore expressible in the form $r = f(z) \cdot g(\theta)$. Clearly the C_{γ_θ} differ only by a linear transformation.

We next prove that C_γ is an ellipse.[5] For this we need to know that there exists a linear orientation-preserving transformation sending C_{γ_0} into itself and p into any other point q of C_{γ_0}.

Let r be the point of tangency of a plane parallel to the x-z plane having a positive y component. Suppose γ_1 is the plane defined by p, o, r. We have shown that C_{γ_0} goes into C_{γ_1} by a linear transformation which leaves invariant points of the z axis. We can repeat the above construction about the line $\overline{or}$. Hence if γ_2 is the plane defined

[5] The remainder of the proof is similar to an argument used by Garrett Birkhoff, Duke Mathematical Journal, vol. 1 (1935), pp. 169–172, Theorem 1.

by q, o, r, then C_{γ_1} goes into C_{γ_2} and p goes into q by a linear transformation leaving the x-z plane and points of $\overline{or}$ invariant. Repeating the above construction about the line $\overline{oq}$, C_{γ_2} goes into C_{γ_0} by a linear transformation which leaves points of $\overline{oq}$ invariant. The product of these transformations is the desired linear transformation.

The set C_{γ_0} is compact and bounded away from o. Therefore the group of all orientation-preserving linear transformations of C_{γ_0} into itself is bounded and hence equivalent, after a linear transformation, to a subgroup G of the orthogonal group [6, p. 465, Theorem 19]. Since G is transitive on lines through o, G must be the entire orthogonal group. The set of points invariant under G is the circle. Therefore a suitable linear transformation sends C_{γ_0} into the circle. It follows that all C_γ through p are ellipses. p was chosen arbitrarily. All C_γ are therefore ellipses. If we now take a particular C_γ and choose its major axis to be the z axis of our construction, S' will be generated by this ellipse tracing an ellipse in the x-y plane and rotating about the z axis. S is therefore an ellipsoid.

References

1. S. Kakutani, *Some characterizations of Euclidean space*, Japanese Journal of Mathematics, vol. 16 (1939), pp. 93–97.

2. W. Blaschke, *Kreis und Kugel*, Leipzig, 1916, pp. 157–159.

3. S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932.

4. H. Löwig, Acta Szeged, vol. 7 (1934), pp. 1–33.

5. P. Jordan and J. von Neumann, *On inner products in linear metric spaces*, Annals of Mathematics, (2), vol. 36 (1935), pp. 719–723.

6. J. von Neumann, *Almost periodic functions in a group* I, Transactions of this Society, vol. 36 (1934), pp. 445–492.

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