ABSTRACTS OF PAPERS

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

394. Leon Alaoglu, Garrett Birkhoff, and B. J. Pettis: *General ergodic theorems.* II. Preliminary report.

As a further application of the methods of the paper *General erogdic theorems,* Annals of Mathematics, (2), vol. 41 (1940), pp. 293–309, the following theorem is proved. The space E is a Banach space which is also a complete linear lattice; furthermore $\| \ |x| \ \| = \|x\|$, and the norm is a monotone non-decreasing function of positive elements. The abelian semi-group G consists of positive linear transformations on E to E; means of transformations of G are denoted by τ. Then, if $\lim \sup_\tau |\tau x| < +\infty$ for all x of E, the limit $\lim_\tau \tau x$ exists in order for all x for which it exists in norm, and the two are equal. These limits of means are defined on page 296 of the above paper. If E is reflexive, the hypothesis of lattice completeness can be dropped, and the limit in norm exists for all x. (Received June 19, 1940.)

395. M. M. Day: *Ordered sets and closures.*

A closure function on X is any function c defined on all subsets of X whose values are subsets of X; c is monotone if $Y \subset Y' \subset X$ implies $cY \subset cY'$. A set A is called effectively ordered if the order relation is transitive and if every element of A has a successor. A natural definition of convergence in a neighborhood space is given which reduces to the definition of the Moore-Smith limit if A is directed. Definitions of closure in terms of neighborhoods and of convergence of functions on effectively ordered sets are given and it is shown that the closures defined by either method are the monotone closures. Tukey's definition of order among directed sets is extended to effectively ordered sets and it is shown that much but not all of the theory that holds for directed sets carries over to effectively ordered sets. (Received June 8, 1940.)

396. M. M. Day: *Reflexive Banach spaces which cannot be made uniformly convex.*

The space $B = (B_1 \times B_2 \times \cdots)_p$ (S. Banach, *Théorie des Opérations Linéaires,* Warsaw, 1932, p. 243) is shown to be reflexive if all B_i are. If B_i is taken to be the i-dimensional space with $\|b\|_{B_i} = \sup |b_j|$ or $= \sum |b_j|, j \le i$, where $b = (b_1, \cdots, b_i)$, it is shown that B is not isomorphic to any uniformly convex space. By choosing subspaces isomorphic to these it can be shown, for example, that if $B_i = L^{q_i}$ or l^{q_i}, $1 < q_i < \infty$, and if there do not exist m and M with $1 < m \le q_i \le M < \infty$ for all i, then B is not isomorphic to a uniformly convex space. (Received June 10, 1940.)

397. H. L. Garabedian, H. S. Wall, and Einar Hille: *Hausdorff means and integral equations.*

If $\{a_n\}$, $\{b_n\}$ are two regular Hausdroff sequences, then $\{b_n\}$ is said to be a divisor of $\{a_n\}$ if there exists a regular Hausdorff sequence $\{c_n\}$ such that $a_n = b_n c_n$, $(n=0, 1, 2, \cdots)$. Let $f_a(x) = \sum a_n(-x)^n$, $f_b(x) = \sum b_n(-x)^n$. The authors show that $\{b_n\}$ is a divisor of $\{a_n\}$ if and only if there exists a regular mass function $\phi_c(u)$ such that $f_a(x) = \int_0^1 f_b(ux)d\phi_c(u)$, $|x| < 1$. By means of the Stieltjes inversion formula they show that this integral equation can be transformed into the equation of R. Schmidt, namely: $\phi_a(u) = \int_0^1 \phi_b(u/v)d\phi_c(v)$ (which holds for all except possibly a finite or countable set of values of u), and obtain in this way a theorem of Hille and Tamarkin (Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 573–577). Application is made to prove the inclusion relation $(H_m, \alpha) \supset (H, \alpha+m-1)$, $(\alpha>0, m=2, 3, 4, \cdots)$, where (H_m, α) denotes summability defined by the mth row in the difference matrix in which the base sequence is the sequence $\{(n+1)^{-\alpha}\}$ defining summability (H, α). (Received June 6, 1940.)

398. Edward Kasner and J. J. De Cicco: *Equilong and conformal transformations of period two.*

The main problem of this paper is to find all equilong transformations of period two. In equilong geometry the set of all equilong transformations of period two may be classified into three distinct types: (T_1) equilong involutions, (T_2) K symmetries, and (T_3) D inversions. This is in contrast with the conformal theory where Kasner has proved that the set of all conformal transformations of period two consists of two distinct types: $(\mathfrak{C}_1)$ conformal involutions; and $(\mathfrak{C}_2)$ conformal symmetries. There are thus five distinct types of transformations of period two; three equilong and two conformal. The equilong transformations of period two may be reduced equilongly to the canonical forms: (T_1) symmetry in the positive y_1-axis $Z = -z$ where z represents the dual variable $x+jy$ with $j^2=0$ and (x, y) the hessian or equilong coordinates of a line, (T_2) symmetry in the origin accompanied by reversal of orientation $Z=\bar{z}$, and (T_3) symmetry in the x_1-axis accompanied by reversal of orientation $Z = -\bar{z}$. It is recalled that the conformal transformations of period two may be changed conformally to $(\mathfrak{C}_1)$ symmetry in the origin $Z_1 = -z_1$ where z_1 represents the complex variable x_1+iy_1 with $i^2 = -1$ and (x_1, y_1) the cartesian coordinates of a point, and $(\mathfrak{C}_2)$ symmetry in the x_1-axis, $Z_1 = \bar{z}_1$. (Received June 5, 1940.)

399. Otto Szász: *On the jump of almost periodic functions and Fourier integrals.*

In a previous paper (Duke Mathematical Journal, vol. 4 (1938)) the author proved a theorem expressing the generalized jump of a periodic function $f(x)$ with period 2π as the limit of $(\pi/\log 2)\{\bar{\sigma}_{2n}(x)-\bar{\sigma}_n(x)\}$, as $n \to \infty$, where $\bar{\sigma}_n$ is the nth arithmetic mean of the series conjugate to the Fourier series of $f(x)$. Here analogous results are established for functions almost periodic in the sense of Besicovitch, and for Fourier integrals of functions in L^p, $1 \leq p \leq 2$. (Received June 8, 1940.)

400. G. A. Whetstone and C. M. Cramlet: *On the invariance of the passivity conditions of systems of partial differential equations.*

The system of r linear differential equations in one dependent and n independent variables $X_i u \equiv a_i \partial u/\partial x_\rho \equiv 0, i=1, \cdots, r; \rho=1, \cdots, n$, is said to be complete if the Pois-

son operator $(X_i X_j)$ applied to u yields a linear combination of the $X_i u$. Under arbitrary analytic transformations on the independent variables the property of completeness is invariant. A well known extension of this result is the theorem: The property that a system be involutory is invariant. The authors have extended these theorems to systems of equations with more than one dependent variable. In terms of a canonical system C which is in orthonomic form and assumed to be passive the general theorem which includes the foregoing classical theorems is: When the system C is transformed by an arbitrary analytic transformation on the independent variables and the new system reduced to an orthonomic canonical form $\bar{C}$, this new system will be passive. (Received June 17, 1940.)

401. Stefan Bergman: *Boundary values of functions satisfying partial differential equations of elliptic type.*

Let $U(z, \bar{z})$, $z = x + iy$, $\bar{z} = x - iy$, be regular in $|z| < 2$ and let it satisfy there the equation $L(U) = U_{z\bar{z}} + \operatorname{Re}(A U_z) + C U = 0$, where A is a complex function of z, $\bar{z}$ regular in $|z| \leqq 2$, C a real function regular in the same region, $U_z = \partial U/\partial x - i\partial U/\partial y$. Under these circumstances $\sum_{n=1}^{\infty} |a_n|^2 n^{-1} < \infty$ is a sufficient condition that U has boundary values almost everywhere on $|z| = 2$ for radial approach. Here $a_n = \int_0^{2\pi} [\tilde{U}(\rho, \phi) K_1^{(n)}(\rho, \phi) + \tilde{U}_\rho(\rho, \phi) K_2^{(n)}(\rho, \phi)] d\phi$ where $\tilde{U}(\rho, \phi) = U(\rho e^{i\phi}, \rho e^{-i\phi})$, $\tilde{U}_\rho(\rho, \phi) = \partial \tilde{U}/\partial \rho$, $0 < \rho < 2$ arbitrary. $K_k^{(n)}(\rho, \phi)$ are certain functions depending only on L and connected in a simple way with the solution $E = V(z, \bar{z}, \zeta, \bar{\zeta}) \lg |z - \zeta| + W(z, \bar{z}, \zeta, \bar{\zeta})$ of $M(E) = 0, z \neq \zeta$, M denoting the equation adjoint to L. (See Sommerfeld, Enzyklopädie der mathematischen Wissenschaften, II A, 7 c, p. 515.) (Received July 27, 1940.)

402. Stefan Bergman and D. C. Spencer: *On distortion in pseudoconformal mapping.*

In the theory of conformal mapping, theorems concerning the change of measure of various euclidean quantities play an important role. For example, the area of certain sub-domains, as the theorems of Golusin and Bermant show, cannot be too small (G. M. Golusin, Comptes Rendus de l'Académie des Sciences de l'URSS, 1937, pp. 617–619; A. Bermant, Comptes Rendus de l'Académie des Sciences de l'URSS, 1938, pp. 137–140). This paper contains analogous results for pseudo-conformal mappings. The concept of "B-area" (Stefan Bergman, Rendiconti dell' Accademia Nazionale dei Lincei, vol. 19 (1934), pp. 474–478) is used, as well as a new representation for four-dimensional volume. The main result is an analogue in two complex variables of the theorem of Golusin. (Received August 1, 1940.)

403. R. P. Boas: *A general moment problem.*

Let $\{\phi_n(x)\}$ be a set of normalized functions belonging to L^2 on $(0, 1)$. The object of this paper is to establish conditions on the ϕ_n under which any sequence $\{a_n\}$ of complex constants such that $\sum |a_n|^2 < \infty$ can be represented in the form $a_n = \int_0^1 f(x) \phi_n(x) dx$, where $f(x) \varepsilon L^2$. It is found that a necessary and sufficient condition is that the quadratic form $\sum \sum x_m \bar{x}_n \int_0^1 \phi_m(x) \bar{\phi}_n(x) dx$ should have a positive lower bound. When the ϕ_n are orthogonal, the condition is trivially satisfied, and the theorem reduces to the Riesz-Fischer theorem. Generalizations of the Hausdorff-Young-Riesz theorems are also obtained, and the theory is applied to some specific sets of functions. (Received July 29, 1940.)

404. R. P. Boas and D. V. Widder: *Functions with positive differences*.

Notation: $\Delta_\delta^0 f(x) = f(x)$, $\Delta_\delta^n f(x) = \Delta_\delta^{n-1} f(x+\delta) - \Delta_\delta^{n-1} f(x)$ $(n=1, 2, \cdots)$. A simplified proof is given of a theorem of T. Popoviciu (Mathematica, Cluj, vol. 8 (1934), pp. 1–85) that if $f(x)$ is continuous in (a, b) and if, for some $n \geq 2$, $\Delta_\delta^n f(x) \geq 0$ for all x and $\delta (>0)$ such that x and $x+n\delta$ are in (a, b), then $f(x)$ is of class C^{n-2} in (a, b) and has continuous right- and left-hand $(n-1)$th derivatives. It can be deduced from results of S. Bernstein that if $\Delta_\delta^n f(x) \geq 0$ for all integers n in a sequence $\{n_k\}$ such that n_{k+1}/n_k is bounded, $f(x)$ is analytic in (a, b); an independent proof is given of this theorem. It is shown that if n_{k+1}/n_k is unbounded a function $f(x)$ can be constructed, of class C^∞, such that $f^{(n_k)}(x) > 0$ $(k=1, 2, \cdots)$, while $f(x)$ is not analytic. (Received July 31, 1940.)

405. A. T. Brauer: *On a property of k consecutive integers*.

Pillai has just proved the following theorem (Proceedings of the Indian Academy of Sciences, Section A, vol. 11 (1940), pp. 6–12): In every set of less than 17 consecutive integers there exists at least one integer which is relatively prime to all the others; there are sequences of k integers for $k=17, 18, \cdots, 430$, however, which have not this property. Pillai conjectures that the same is valid for every $k \geq 17$. It is proved that this conjecture is true. (Received July 29, 1940.)

406. Herbert Busemann and Walther Mayer: *On the foundations of calculus of variations*.

The minimizing arcs of a two-dimensional variational problem with fixed end points need not be differentiable everywhere if the integrand $F(x, y; \dot{x}, \dot{y})$ is regular and continuous in all four variables. But if $F(x, y; \dot{x}, \dot{y})$ is regular and continuous and satisfies a Lipschitz condition with respect to x and y, all minimizing arcs are continuously differentiable. However, in general line elements will occur which are contained in no or in infinitely many minimizing arcs. (Received July 30, 1940.)

407. R. H. Cole: *The expansion problem associated with an ordinary linear differential equation and boundary conditions applying at a set of collinear points*.

The expansion problem associated with the differential equation $u^{[n]} + P_1 u^{[n-1]} + \cdots + P_n u = 0$ whose solutions are conditioned relative to an arbitrarily given finite set of points on a fundamental interval by n linear boundary relations is a familiar one. It was studied by Wilder (Transactions of this Society, vol. 18 (1917)) who restricted his investigation to the equation in which the parameter λ appears only in the coefficient function P_n in the form $P_n = P_{n,0}(x) + \lambda^n$, and to boundary conditions which are free from λ. The present paper deals with the question of the convergence of the expansion when the functions P_k are taken as polynomials in λ with coefficients which are functions of x, and when the coefficients of the boundary relations are arbitrary polynomials in λ. The reduction of the given system to matrix form permits the adoption of the formal expansion of an arbitrary vector (that is, a set of n arbitrary functions) developed by Langer (Transactions of this Society, vol. 46 (1939)) for problems of this type. His convergence theorem, however, is presented under regularity restrictions which are not fulfilled by the problem at hand. A con-

vergence theorem is developed here which shows the uniform convergence of the simultaneous expansion of the n arbitrary functions under appropriate regularity conditions. (Received July 5, 1940.)

408. H. V. Craig: *On extensors*.

The first part of this paper is concerned with the transformation equations of certain quantities called "crossed extensors." The distinguishing feature of these equations is that the summations are not restricted to adjacent Greek and Latin indices. The second part contains certain applications of extensor methods to classical differential geometry. One of the results is such that it can be outlined briefly. Let ρ be the radius vector from a given origin to a point with coordinates x and let $x = x(t)$ be the equations of a parametrized arc. Obviously, ρ sup (M), the Mth derivative of ρ with respect to t, is a vector function of x sup $(\alpha)a, \alpha = 0, \cdots, M$. The partial derivatives ρ sup (M) inf $(\alpha)a$ evidently transform cogrediently to an excovariant extensor. Consequently the scalar products of these vectors, denoted by g inf $\alpha a \beta b$, are the components of an extensor. Therefore the contraction of the metric tensor g sup bc with g inf αa Mc is a tensor-extensor which is represented by ρ sup b inf αa. The contraction of this quantity with V sup $a(\alpha)$, the derivatives of a contravariant vector V, over the reduced range $M - \beta$ to M, is the βth intrinsic derivative of V multiplied by a constant. From the extensor point of view, the tensor character of intrinsic differentiation is obvious without knowledge of the affine connection. (Received July 31, 1940.)

409. H. B. Curry: *A revision of the fundamental rules of combinatory logic*.

This paper is concerned with some general theorems concerning improvements in the formulation of the ultimate foundations of the theory of combinators. (Cf. the first part of the author's thesis, American Journal of Mathematics, vol. 52 (1930), pp. 509–525.) For the most part these improvements were suggested by the work of Rosser (cf. his thesis, Annals of Mathematics, (2), vol. 36 (1935); pp. 127–139) who formulated a weakened system of combinatory logic in which the rules had a simple character not possessed by the rules of the original system. This paper discusses ways of formulating the original system which are analogous to Rosser's formulation. A formulation is given based on the primitives of the original system, and also in terms of the Schönfinkel primitives, S and K. (Received July 27, 1940.)

410. H. B. Curry: *Consistency and completeness of the theory of combinators*.

Some theorems relating to the consistency and completeness of the theory of combinators were proved in the author's thesis (American Journal of Mathematics, vol. 52 (1930), pp. 509–536, 789–837). Later, analogous theorems were proved by Rosser (Annals of Mathematics, (2), vol. 36 (1935), pp. 127–150 and Duke Mathematical Journal, vol. 1 (1935), pp. 328–355); his theorems were, in some respects, more powerful, but they applied only to a weakened system. In this note it is shown that the strongest theorems can be extended to the full theory of combinators, with or without a normality condition similar to Rosser's. The methods involve some slight generalization of the methods used by Church and Rosser, including their theorem of conversion (Transactions of this Society, vol. 39 (1936), pp. 472–482). (Received July 30, 1940.)

411. John Dyer-Bennet: *A note on partitions of the set of positive integers.*

The partitions of the set of positive integers which are homomorphic with respect to addition are classified, and it is noted that they are also homomorphic with respect to multiplication. A necessary and sufficient condition that such a partition be homomorphic with respect to exponentiation is given. Elementary number-theoretic methods are used. (Received July 2, 1940.)

412. Benjamin Epstein: *A contribution to the theory of two complex variables in certain domains.*

In this paper the author investigates a.f. of 2 c.v. (analytic functions of two complex variables) in domains $\mathfrak{S}^4 = \sum_{|\gamma| \le 1} S^2[\theta_1(\gamma), \theta_2(\gamma), \infty]$, where $\mathfrak{S}^4$ is a domain of four-dimensional space which is the set sum of sectors $S^2[\theta_1(\gamma), \theta_2(\gamma), \infty]$ lying in the plane $z_2 = \gamma$ and with boundary made up of two rays proceeding from the point $z_1 = 0$ and making angles $\theta_1(\gamma), \theta_2(\gamma)$ respectively with the positive real axis. The following result is proved: Let $\mathfrak{g}^1$ be a point set of $|z_2| = 1$ which has the property that the intersection of $\mathfrak{g}^1$ with every arc of $|z_2| = 1$ of length Γ ($\Gamma \ge \Gamma_0$, Γ_0 depending only on $\mathfrak{S}^4$) contains a set of measure equal to or greater than $\Gamma/p (p \le p_0 < \infty)$; let $f(z_1, z_2)$ be a bounded a.f. of 2 c.v. regular in $\mathfrak{S}^4 - \mathrm{E}[z_1 = 0, |z_2| \le 1]$; and let $\underline{\mathrm{Bd}}_{z_1 \to 0} f(z_1, e^{i\phi_2}) = 0$, $e^{i\phi_2} \varepsilon \mathfrak{g}^1$, uniformly in ϕ_2. Then $\lim_{z_1 \to 0} f(z_1, z_2) = 0$, $|z_2| < 1$. Other theorems of similar type can be proved. Investigations concerning a.f. of 2 c.v. in the domains $\mathfrak{S}^4$ have many applications, since it is often possible to reduce the study of a.f. of 2 c.v. in an arbitrary domain whose boundary hypersurface contains a segment of an analytic surface to the study of a.f. of 2 c.v. in a $\mathfrak{S}^4$-domain (cf. Bergmann, Journal für die reine und angewandte Mathematik, vol. 169 (1933), pp. 1–42). (Received July 30, 1940.)

413. G. C. Evans: *Surfaces of minimum capacity.*

The author considers the problem of finding for a given closed curve in space the surface cap which is of minimum electric capacity. For this it is assumed that the curve itself is of zero capacity, and that a sphere containing it can be mapped in a continuous one-one manner on itself in such a way that the curve is represented by a circle. A surface is found which satisfies the minimal condition by determining a certain two-valued harmonic function on a Riemannian manifold of two space sheets. The surface is a level surface of this function. It furnishes the minimum capacity among all surfaces which are made up of sufficiently smooth pieces and have the given curve as sole boundary. (Received July 29, 1940.)

414. H. T. Fleddermann: *The equality of measure functions.*

Besicovitch (Mathematische Annalen, vol. 98 (1927), p. 422) defines the density of a linearly measurable plane set at a point of it. In this definition he makes use of Carathéodory's measure. In this note, the definition is extended by replacing the Carathéodory measure by the Gross measure and by any general measure function. It is shown that if the density obtained by using the Gross measure is equal to 1, everywhere, then the density obtained with the Carathéodory measure is equal to 1, almost everywhere, and vice versa. It is further proved that under these conditions the Gross and Carathéodory measure of the set are equal. It follows from this and the fact that the Gross measure is a minimum measure that the density obtained by using

any general measure function lying between the Gross and Carathéodory functions is also equal to 1 and the measure of the set is the same as the Gross or Carathéodory measure. As the Carathéodory measure is used as an upper bound, a class of measure functions is obtained for which it is a maximum. (Received July 30, 1940.)

415. J. S. Frame: *The double cosets of a finite group.*

Let H be a subgroup of order h of a finite group G of order $g = nh$. When the permutation group G_H of degree n on the right cosets is written as a group of matrices and completely reduced, let the irreducible representations Γ_i of degree n_i, $i = 1, 2, \cdots, r'$, appear with multiplicity μ_i as components of G_H, and let $c_i = 1$, -1, or 0 according as Γ_i has a symmetric, an alternating, or no bilinear invariant. Let G contain r double cosets $K_t = HS_tH/d_t$, $t = 1, 2, \cdots, r$, consisting respectively of $k_t = h/d_t$ right cosets HS_t which are permuted transitively among themselves when multiplied on the right by elements of H. Let N_H of these be self-inverse. Then after some preliminary theorems about the distribution of elements within the double cosets of G, two principal theorems are proved. The first, which generalizes a result proved by the author in a previous paper (Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 132–139), states that $N_H = \sum_{i=1}^{r'} c_i \mu_i$. The second, which proves and generalizes a theorem conjectured by the author in another paper (Duke Mathematical Journal, vol. 3 (1937), pp. 8–17), states that $n^{r-2} \prod_{t=1}^{r} k_t = (\prod_{i=1}^{r'} n_i^{\nu} \overline{P}_1 P_1$, where $\nu = \mu_1^2$ and P_1 is an algebraic integer in the field of the characters of the components of G_H. (Received July 23, 1940.)

416. K. O. Friedrichs: *On hyperbolic differential operators.*

Uniqueness and existence of the solution of Cauchy's problem for linear hyperbolic differential equations can be based on the projection theorem in Hilbert space and the use of certain "mollifying" integral operators, which had been employed for elliptic differential operators in a previous paper of the author. "Hyperbolic" differential equations are here understood in a wider sense; they are written as a system of the first order and contain the classical wave equations as a special case. The differential operator is to be closed by extending it to a certain linear space of L^2-integrable functions. The values of the solution are prescribed on a spacelike surface; the conditions for these initial data are such that they persist in the process of propagation. (Received July 30, 1940.)

417. Orrin Frink: *Series expansions in linear vector space.*

If the set $P = \{p_n\}$ is closed and minimal in the real or complex Banach space B, it is shown that every element x of B has a uniquely determined series expansion (1) $\sum \xi_n p_n$ in terms of P, called its *minimal expansion*, such that (I) the coefficient of p_n, in any sequence of linear combinations of P which converges to x, converges to ξ_n; (II) the coefficient ξ_n is a continuous additive function of x; (III) P is part of a biorthogonal system $\{p_n, f_{ni}\}$ and conversely, if $\{p_n, f_n\}$ is biorthogonal, then $\{p_n\}$ is minimal; (IV) the partial sums of (1) are in one sense the best possible approximations to x. A condition is given which is sufficient that the series (1) converge absolutely to x. If P is normalized, it is shown that $\xi_n \to 0$. Semiregular summability methods are introduced, and it is shown that no such method sums (1) to the wrong limit, while at least one such method sums (1) to x if only a finite number of the coefficients ξ_n are zero. Applications are given to Schauder series and minimal polynomial series, including power series and Newton interpolation series, both real and complex. (Received July 27, 1940.)

418. Abe Gelbart: *On approximations of functions of two complex variables regular in multiply-connected four-dimensional domains.*

In functions of one complex variable it is shown that every analytic function $f(z)$ regular in a multiply-connected domain D^2 of n holes can be approximated uniformly by $(n+1)$ functions, the first n being rational with one singularity at a point in one of the holes, and the $(n+1)$st being a polynomial. With the aid of Bergman's integral formula for functions of two complex variables (Recueil Mathématique, vol. 43 (1936), pp. 851–862), it is shown that an analytic function of two complex variables, with n singularity surfaces in a four-dimensional region $\mathfrak{M}^4$, can be approximated uniformly in a multiply-connected closed region $\overline{\mathfrak{A}}^4$, interior to $\mathfrak{M}^4$ and excluding the n singularity surfaces, by $[n(n+1)+1]$ functions, the first $n(n+1)$ having just two singularity surfaces and the last being a polynomial. By imposing further conditions on the domain $\overline{\mathfrak{A}}^4$, it is shown that each of the $n(n+1)$ functions can be approximated uniformly in $\mathfrak{A}^4$ by rational functions that have the same two singularity surfaces. (Received July 30, 1940.)

419. Louis Green: *Twisted cubics associated with a space curve.* II.

This paper is an extension of results of the author (American Journal of Mathematics, vol. 62 (1940), pp. 285–306) on local projective properties of a space curve Γ. It is found that Γ possesses at each of its points a two-parameter family of five-point cubics which are intimately related to the osculating quadrics of Γ. By means of these cubics, new configurations related to Γ are obtained and new characterizations are given to some known ones. A number of the cubics are found to have special properties of interest. (Received July 31, 1940.)

420. D. W. Hall and W. T. Puckett: *Arc-preserving transformations of a certain class of spaces.*

It is shown that the class of cyclic strongly arcwise connected continua (W. T. Puckett, this Bulletin, abstract 46-5-316) consists exactly of all cyclic locally connected continua A such that every arc-preserving transformation $T(A) = B$, where B is not an arc, is topological. (Received July 29, 1940.)

421. Marshall Hall: *Projective planes.*

An investigation of projective planes, especially non-Desarguesian types. Consideration of the generation of planes, particularly "free" planes. A study is made of projections, collineations, and universal configurations. (Received July 22, 1940.)

422. P. R. Halmos and R. A. Leibler: *Square roots of measure preserving transformations.*

Let T be a one-to-one, measure preserving (not necessarily continuous) transformation of the m-dimensional unit cube into itself. The problem of the present paper is to find conditions necessary and sufficient for the existence of another such transformation S, called a square root of T, for which $S^2 = T$. A complete solution is given in case T has a pure point spectrum: in other words, in case there exists a sequence of numbers α_n and a complete orthonormal set of functions $f_n(x)$ in L_2 such that $f_n(Tx) = \alpha_n f_n(x)$. The following two theorems (indicating, respectively, the nature of the necessary and of the sufficient conditions) are typical. I. If T has a square root, then the multiplicity of -1 with respect to the point spectrum of T is even (is

$0, 2, 4, \cdots, \infty$). II. If T is metrically transitive and has a pure point spectrum, and if -1 does not belong to the spectrum of T, then T has a square root. (Received July 30, 1940.)

423. J. G. Herriot: *Nörlund summability of double Fourier series.* Preliminary report.

Grünwald (Proceedings of the Cambridge Philosophical Society, vol. 35 (1939), pp. 343–350) has proved that the square partial sums s_{nn} of the double Fourier series of an integrable periodic function of two variables are summable $(C, 1)$ to the value of the function at any point of continuity. The author replaces the $(C, 1)$ method by a Nörlund method with nonnegative, nonincreasing coefficients and obtains necessary and sufficient conditions that the above result remains true. The necessity is proved by appealing to a theorem of Banach and Steinhaus on linear functionals. The sufficiency follows from Grünwald's result. In particular, it follows that the (C, α) method applied to square partial sums possesses the localization property if and only if $\alpha \geqq 1$. Again consider the Nörlund transforms t_{mn} of the partial sums s_{mn} obtained by a double Nörlund transformation $\{p_m q_n\}$, p_n and q_n being nonnegative and nonincreasing. Consider the limit of t_{mn} when $m, n \to \infty$ so that $m/n \leqq \lambda, n/m \leqq \lambda$ ($\lambda \geqq 1$, a fixed number). Necessary and sufficient conditions that this method of summability should possess the localization property are obtained. In particular, when $0 < \alpha$, $\beta \leqq 1$, (C, α, β) as above restricted possesses the localization property if and only if $\alpha = \beta = 1$. (Received July 25, 1940.)

424. Harold Hotelling: *Experimental determination of the maximum of an empirical function.*

In physical and economic experimentation to determine the maximum of an unknown function, for example, of a monopolist's profit as a function of price or of the magnetic permeability of an alloy as a function of its composition, the characteristic procedure is to perform experiments with chosen values of the argument x, each of which then yields an observation, subject to error, on the corresponding functional value $y = f(x)$. The values of x need, however, to be chosen on the basis of earlier experiments in order to make the determination efficient. The experimentation properly proceeds, therefore, in successive stages, with the values used at each stage determined with the help of the earlier work. The question, what distribution of x as a function of previous results should be used, is discussed in this paper on the basis of various hypotheses regarding the function, and further criteria. In particular, a conflict is shown to exist under some conditions between the criterion of minimum sampling variance and that calling for absence of bias. (Received August 1, 1940.)

425. Fritz John: *Discontinuous convex solutions of difference equations.*

This paper is a supplement to the author's paper (Acta Mathematica, vol. 71 (1939), pp. 175–189) which dealt only with continuous convex solutions. It includes as special cases results by A. E. Mayer (Acta Mathematica, vol. 70 (1938), pp. 57–62) and by H. P. Thielman (this Bulletin, vol. 46 (1940), p. 432). It is shown that a difference equation can have only a discontinuous convex solution if it has more than one continuous convex solution. A difference equation $\sum_k a_k f(x+k) = 0$ with constant a_k has a discontinuous convex solution when and only when the equation $\sum_k a_k x^k = 0$ has a positive real root. (Received July 25, 1940.)

426. Mark Kac: *On a question of Ramanujan.*

Let $d(n)$ denote the number of divisors of n. Ramanujan conjectured (*Collected Papers*, p. 333, Question 770) that the series $\sum_{k=1}^{\infty}(-1)^k[d(2k+1)]/(2k+1)$ is convergent. The present paper furnishes a short and simple proof of this statement. It turns out that the sum of the above series is $\pi^2/16$. (Received July 27, 1940.)

427. W. H. McEwen: *On the degree of convergence of a series associated with an integro-differential system.*

This paper deals with the expansion of a function $f(x)$ in terms of characteristic solutions of a given homogeneous integro-differential system involving a parameter. Denoting the partial sums of the series by $S_N(x)$ ($N=1, 2, \cdots$), and letting m denote an arbitrary positive integer, hypotheses are obtained under which $f^{(k)}-S_N{}^{(k)}=0$ $(1/N^{m-k})$ uniformly on the interval of x in question, where $k=0, 1, \cdots, m-1$. (Received July 31, 1940.)

428. H. E. Newell: *On the asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain.* Preliminary report.

The matric differential equation $(d/dx)Y(x,\lambda)^{n \times n} = \{\lambda(\delta_{ij}r_j(x))^{n \times n}+(q_{ij}(x,\lambda))^{n \times n}\}$ $\cdot Y(x, \lambda)^{n \times n}$ is discussed for the parameter λ large in absolute value, and for x in a finite simply connected region of the complex plane, within which the coefficient functions are analytic with the differences $r_i(x)-r_j(x)$, $(i \neq j)$, not zero, and on the boundary of which the coefficient functions may have poles, and the differences $r_i(x)-r_j(x)$, $(i \neq j)$, may have zeros. Under four distinct sets of conditions, three of which admit of meromorphic $q_{ij}(x, \lambda)$, regions of existence abutting a specified pole of an $r_j(x)$ or a specified zero of a difference $r_i(x)-r_j(x)$, $(i \neq j)$, are established for a solution of the form $P(x, \lambda)(\delta_{ij} \exp \{\lambda \int^x r_i(x)dx\})$, where $P(x, \lambda)$ reduces uniformly in x to the identity matrix when λ becomes infinite. Two of the sets of conditions are also sufficient that $P(x, \lambda)$ admit, for $|\lambda|$ large, of an asymptotic series expansion in λ^{-1}. The method employed is similar to that recently used by R. E. Langer in treating the case in which the coefficient functions are analytic with the differences $r_i(x)-r_j(x)$, $(i \neq j)$, bounded from zero. Finally, it is also shown that the results of both the present and Langer's papers apply to infinite as well as to finite x regions. (Received July 3, 1940.)

429. Rufus Oldenburger: *La teoria de los polinomios de orden superior.*

This paper will be published in Spanish in the scientific and engineering publication "Ingenieria" of the National University of Mexico, where it was given as a series of five lectures. In this paper, solutions are obtained for all of the open questions listed in the last paragraph of the author's article on polynomials (Annals of Mathematics, (2), vol. 41 (1940), p. 709), together with applications. (Received July 29, 1940.)

430. J. M. H. Olmsted: *Lebesgue theory on a Boolean algebra.* Preliminary report.

Any partially ordered linear space with an element 1 (cf. H. Freudenthal, Proceedings, Akademie van Wetenschappen, Amsterdam, vol. 39 (1936), p. 641) can

be extended to a commutative ring, the multiplicative unit of which is the element 1. This extension is the space of all functions, from real numbers to elements of the underlying Boolean algebra, decreasing monotonely from the one element of the algebra to the zero element. Such functions are analyzed for an arbitrary Boolean algebra with countable superior and inferior. Furthermore, multiplication of functions and the existence of a measure on the Boolean algebra give rise to a theory of the L^p spaces. The Nikodym theorem for absolutely continuous functions is again available. (Received July 8, 1940.)

431. F. W. Perkins: *Polygonal and polyhedral means of functions.* Preliminary report.

Let R be a finite region of euclidean space of three dimensions. Let $G_P(a_P)$ be a closed subregion of R bounded by a regular polyhedron with center at P and radius a_P, and let $G_P(\rho)$ be the closed subregion bounded by a concentric similar polyhedron of radius $\rho \leq a_P$, similarly oriented. Let $U(P)$ be a function of P, real and continuous in R. The author considers (as functions of P and ρ) various means of u associated with $G_P(\rho)$, especially the arithmetic means of u on the vertices, on the radial lines drawn to the vertices, on the triangular areas bounded by these radial lines and the edges, on the faces, and over the volume. Various relations involving these means are established, some of which are of particular interest in the theory of harmonic and subharmonic functions. Similar methods can be used in the plane; here some parts of the theory are related to topics studied by Frazer (*On the moduli of regular functions*, Proceedings of the London Mathematical Society, vol. 36 (1934), pp. 532–546). (Received July 25, 1940.)

432. G. Pólya: *Sur l'existence de fonctions entières satisfaisant à certaines conditions linéaires.*

The present paper deals with an interpolation problem for entire functions. Given two sequences of complex numbers $\{a_n\}$ and $\{A_n\}$ and a sequence of nonnegative integers $\{\alpha_n\}$. Does there exist an entire function $F(z)$ such that $F^{(\alpha_n)}(a_n) = A_n$ for all n? The author shows the existence of a solution when the sequence $\{a_n\}$ is periodic of period p, $\alpha_{n+p} = \alpha_n + p$, $\alpha_n \leq \alpha_{n+1}$, $\lim |A_n/\alpha_n!|^{1/\alpha_n} = 0$, and there is no polynomial of degree less than p satisfying the first p homogeneous equations. (Received July 20, 1940.)

433. E. J. Purcell: *Space Cremona transformations of order* $m+n-1$.

Consider a curve C_n of order n having $n-1$ points on each of two skew lines d and d', and a curve C_m' of order m having $m-1$ points on each of d and d' (m, n any integers). A generic point P determines a ray through it intersecting C_n once in α and d once in β. P also determines a ray through it intersecting C_m' once in γ and d' once in δ. Lines $\alpha\delta$ and $\beta\gamma$ intersect in P', the correspondent of P in a Cremona transformation of order $m+n-1$. Special positions of the defining curves give rise to involutions. The paper will appear soon in this Bulletin. (Received July 29, 1940.)

434. J. F. Ritt: *On a type of algebraic differential manifold.*

For manifolds of systems of differential equations involving one unknown function, operations of addition, multiplication and differentiation are defined and in-

vestigated. The paper will appear in an early number of the Transactions of this Society. (Received June 27, 1940.)

435. L. B. Robinson: *Singular solutions of nonlinear functional equations.*

Little is known about nonlinear functional equations. The author has computed some singular solutions and gives a simple specimen. Given the function $u(x) = x^\lambda/c$. This function satisfies an equation of the type $u'(x) = P(x)\{u(x^m)\}^2$ since it satisfies (I) $u'(x) = \lambda c x^{-2m\lambda+\lambda-1}\{u(x^m)\}^2$ and the transformation $x = 1/\xi$ sends this into (II) $w'(\xi) = \lambda c \xi^{+2m\lambda-\lambda-1}\{w(\xi^m)\}^2$. Equation (II) admits a solution holomorphic at $\xi = 0$ and depending on one parameter. It also has the singular solution $w(\xi) = \xi^{-\lambda}/c$. (Received July 25, 1940.)

436. R. M. Robinson: *Stencils for solving $x^2 \equiv a \pmod{m}$.*

A set of 272 stencils for solving $x^2 \equiv a \pmod{m}$ by the method of exclusion has been constructed on Hollerith cards. This provides a practical method for solving such congruences; the solution is particularly simple if $m < 3000$. (Received July 29, 1940.)

437. E. Rosenthall: *On the diophantine equation $x^3 + y^3 = z^3 + w^3$.*

The letters A, B, C, denote arbitrary algebraic integers of the quadratic number field $K(\rho)$, ρ an imaginary cube root of unity; all other letters represent rational integers. The conjugate of any integer Q is denoted by $\bar{Q}$. The following theorem is proved: All rational integer solutions of the equation $x^3 + y^3 = z^3 + w^3$ are given by $x = -p(a+b)/q$, $y = p(a-2b)/q$, $z = -p(c+d)/q$, $w = p(c-2d)/q$ where $a + b\rho = (\overline{AB} \cdot B\bar{B} - AC \cdot C\bar{C})A\bar{C}$, $c + d\rho = (\overline{AB} \cdot B\bar{B} - AC \cdot C\bar{C})AB$ and it suffices to take q equal to the G.C.D. of e, f, g, where $\overline{AB} \cdot B\bar{B} - AC \cdot C\bar{C} = e + f\rho$ and $\overline{AC} \cdot \overline{AB} - AC \cdot AB = g + 2g\rho$. The proof is obtained by operating on a multiplicative equation in $K(\rho)$. Complete rational integer solutions are also obtained for each of the following equations: $x^3 + y^3 + z^3 - 3xyz = u^3 + v^3 + w^3 - 3uvw$ and $x^3 + y^3 + z^3 - 3xyz = u^3$. (Received June 29, 1940.)

438. A. R. Schweitzer: *Concerning general abstract relational spaces. II.*

In continuation of a paper reported in this Bulletin (abstract 46-3-190) the author constructs postulates for relational spaces $S_{n+1}(G, H)$ ($n = 1, 2, 3, \cdots$) when G is any substitution group on $n+1$ elements and H is an arbitrarily selected subgroup of G, including G. Distinction is made between "absolute" and "relative" invariance of the generating relation K_H effective between two $(n+1)$-ads of elements; the latter type of invariance is based on correspondence between the elements of the two $(n+1)$-ads. The symmetrical and transitive relation K_H is relatively invariant under G and absolutely invariant under H but not absolutely invariant under any substitution which is not in H. When G is the symmetric group the author's previous theory is obtained. When G is the alternating group and H coincides with G, instances include a system of postulates equivalent to the author's descriptive system ${}^3R_3$ (American Journal of Mathematics, vol. 31 (1909), pp. 387–388). When G is arbitrary the space $S_{n+1}(G, G)$ on $n+1$ elements consists of a definite set of $(n+1)$-ads corresponding to the group G on these elements. Other examples are given. (Received July 30, 1940.)

439. D. J. Struik and Alexander Wundheiler: *Affine representation of systems of forces.*

The well known geometrical representations of systems of forces (Study, Ball, and others) have an arbitrary character. This paper attempts a systematic treatment by introduction of a set of sliding affine figures (vectors, rotors, vector-rotors, rotor-vectors, and so on) obtained by combination of the properties of direction, orientation and normalization. Under the metric group certain figures merge together assuming traditional forms. Coordinates are assigned to these figures on the basis of their relations to projective figures. Merging conditions under the metric group with reflections are studied in particular. (Received August 1, 1940.)

440. H. S. Wall: *Real power series bounded in the unit circle.*

The author investigates the class T of power series $f(x) = \sum c_i(-x)^i$ with real coefficients, radius of convergence greater than or equal to 1, and with the property that $M(f) = \text{l.u.b.}_{|x|<1}|f(x)| \leq 1$. The principal result is the theorem that to each function $f(x)$ of T there corresponds a function $F(x)$ of T which is representable in the form $F(x) = \int_0^1 x d\phi(u)/(1+xu)$ as a Stieltjes integral with respect to a bounded monotone mass function $\phi(u)$, such that $f(x) = [f(0) - x + (1-x)F(z)]/[1 - xf(0) + (1-x)F(z)]$, $z = 4x/(1-x)^2$. As $f(x)$ ranges over T, let $F(x)$ range over the subclass T_0 of T. Then a function is in T_0 if and only if it has a continued fraction representation of the form $(1+\lambda_0)(1-\lambda_1)x/4 + (1+\lambda_1)(1-\lambda_2)x/1 + (1+\lambda_2)(1-\lambda_3)x/4 + \cdots$ (a 4 appears in every other partial denominator), where $-1 \leq \lambda_n \leq +1$, $(n = 0, 1, 2, \cdots)$. The sequence $\{\lambda_n\}$ is the sequence associated with $f(x)$ by the algorithm used by Schur. (Received July 5, 1940.)

441. J. H. C. Whitehead: *On incidence matrices, nuclei and homotopy types.*

W. Hurewicz (Proceedings, Akademie van Wetenschappen, Amsterdam, vol. 39 (1936), p. 125) has asked: "Are two closed manifolds homeomorphic if they are of the same homotopy type?" It appears, among other things, that two closed, polyhedral manifolds (using the word manifold in its narrowest sense), which are of the same homotopy type, need not be combinatorially equivalent, and need not even have the same nucleus (J. H. C. Whitehead, Proceedings of the London Mathematical Society, vol. 45 (1939)). In fact, two lens spaces of types (m, q) and (m, r) are of the same homotopy type provided they have the same intersection invariant, that is, provided $r \equiv \pm l^2 q \pmod{m}$ for some l. This corrects an error in my note *On certain invariants introduced by Reidemeister* (Quarterly Journal of Mathematics (Oxford), vol. 10 (1939)) and it has not even been proved that these are topological invariants. For example, combinatorially inequivalent lens spaces of types (7, 1) and (7, 2) are of the same homotopy type but have different nuclei. Whether or no they are homeomorphic remains an open question. Reidemeister's invariants are the same for polyhedra with the same nucleus. (Received July 5, 1940.)

442. P. M. Whitman: *Splittings of a lattice.*

A "splitting" of a lattice L is defined as a partition of its elements into two "splinters": a principal ideal and a dual of a principal ideal. Necessary and sufficient conditions for the existence of splittings are given. The properties of the "edge" along which the splitting is made are investigated, and conditions given in terms of the

nature of the edge and splinters that L be modular or distributive. Successive splittings form a lattice. The free lattice of n generators has a proper sublattice isomorphic to itself. (Received July 22, 1940.)

443. L. R. Wilcox: *A topology for semi-modular lattices.*

Let L be a semi-modular lattice (cf. Annals of Mathematics, (2), vol. 40 (1939), pp. 490–505) which is atomistic and satisfies the chain conditions. Let the set P of points (or atoms) of L be a metric space. It is shown under two mild assumptions of continuity that the topology of P can be extended to L so that L is a Hausdorff space. Indeed, if δ is the metric of P, and d is the dimension function on L, then neighborhoods of any element $a \; \varepsilon \; L$ are defined in the following manner. Let $\epsilon < 0$, and let $p_1, \cdots, p_k$ be a finite system of independent points whose sum is a; then $U(a; \epsilon; p_1, \cdots, p_k) \equiv [b; \; d(b) = d(a), \; \delta(p_i, b) < \epsilon]$, where $\delta(p_i, b)$ is the distance between the sets $[p_i]$ and $[p; \; p \leq b]$. The purpose of the paper is to clarify and unify the foundations of projective, affine and euclidean differential geometry by lattice-theoretic methods. (Received August 1, 1940.)

444. Orla V. Wood: *On relations between solutions of the differential equation of the second order with four regular singular points.*

The differential equation of the second order with four regular singular points $(0, 1, a, \infty)$ has not been seriously studied, nor has a method been given of extending its integrals from one domain of existence to another. The author considers the difference equation arising from the recurrence relations satisfied by the coefficients in a series solution about the origin. The difference equation is homogeneous, of the second order, the coefficients being of degree zero, two, and four, respectively. The solution of this equation requires the determination of a form suitable to represent it at infinity. An asymptotic solution is found from consideration of a related difference equation which is normalized. Making use of this solution, a Barnes type contour integral is formed. The integral is shown to converge, and to be an analytic solution of the differential equation throughout the complex plane with a cut along the positive real axis. Application of Cauchy's residue theorem to the contour integral leads to an analytic continuation outside the unit circle of the function represented by the original series, and to a relation between solutions of the differential equation. (Received July 30, 1940.)

445. Alexander Wundheiler: *Some basic notions in meta-geometry.*

A figure is a set of points. A figure and a normal divisor of a group G determine a G-hyper-figure of the group G. Two hyper-figures are H-equivalent, H being a subgroup of G, if they can be associated in a way invariant under the group H. A class of H-equivalent hyper-figures determines an H-object under the group H. An H-object has definite coordinates in any coordinate system of the group H. Examples: figure— an attached vector; hyper-figure—a free vector; a free contravariant and a free covariant vector are equivalent under the metric group. (Received August 1, 1940.)