master of lucid exposition, and the book is to be warmly recommended to physicists and mathematicians and to those members of the public who are interested in getting a general view of modern theoretical physics. The book is well turned out, shows only a few trivial misprints (the omission of "h" from "psychological" in the table of contents is the most shocking), and has a much fuller index than those found in other books of this sort (if there are other books of this sort).

J. L. SYNGE

Colloque Consacré à la Théorie des Probabilités. Edited by M. Fréchet and E. Borel. (Actualités Scientifiques et Industrielles, nos. 734–740.) Paris, Hermann.

The proceedings of this colloquium are published in eight small volumes. They comprise an excellent collection of articles which would be an extremely valuable addition to the library of anyone interested in the theory of probability. Although very little of the material is of a purely expository nature, these volumes furnish a rather complete picture of the modern developments of this theory. The following is an outline of the contents of the various conferences.

Volume I. *Conférences d'Introduction et d'Initiation.* 1938

This volume contains two introductory addresses, 1. *Introduction*, by R. Wavre, 51 pages, and 2. *Allocution*, by M. Fréchet. These addresses are followed by:

3. *Les principaux courants dans l'évolution récente des recherches sur le calcul des probabilités*, by M. Fréchet. The author outlines the contributions, trends, and methods of the modern theory of probability. This paper constitutes only the first part of Fréchet's discussion. The remainder appears in Volume II.

4. *Promenade au hasard dans un réseau de rues*, by G. Pólya. The author considers certain probability problems leading to linear partial difference equations of the second order. Limiting cases of these problems admit of physical interpretations. In the limit the difference equations become differential equations. Moreover the solutions of these differential equations with suitable boundary conditions give asymptotic values for the solutions of the difference equations. The author discusses a promenade along a street of infinite length in which the direction of promenade is settled by the tossing of a coin at the end of each block. The corresponding physical problem is the diffusion of a salt solution in a tube. Furthermore the motion of rocks

in a river is related to the above promenade. An n-dimension promenade is also discussed. The reasoning is heuristic and the paper is suggestive of further research.

5. *Wahrscheinlichkeitsaussagen in der Quantentheorie der Wellenfelder*, by W. Heisenberg. The author states that a fundamental problem in modern physics is to ascertain at what points, in the description of natural phenomena, statistical considerations play a deciding role. As an example he considers the emission of electrons from a point source, the electrons being permitted to pass through two slits and strike a nearby plate. This may be regarded as a statistical phenomenon and hence we should expect that the probability of reaching a given point of the plate through one slit plus the probability of reaching this point through the other slit would be equal to the probability of reaching the point through one slit or the other. But this contradicts the fact that the problem can be regarded as a wave phenomenon and hence that there should be interference bands. The author mentions a number of other difficulties which have arisen in modern physics and suggests that the resolution of these difficulties may be accomplished by the introduction of new statistical considerations.

Volume II. *Les Fondements du Calcul des Probabilités*. 1938. 99pp.

This volume constitutes a fascinating debate on the foundations of the theory of probability. A variety of points of view (not only of the participants of the congress but also of those who were unable to attend) are presented and criticized. Frequently answers to criticisms appear. This volume furnishes excellent material for the study of the philosophical aspects of the theory of probability.

1. *Sur la définition des variables éventuelles*, by P. Cantelli. Since Cantelli was unable to attend the congress only an abstract of his paper appears in this series.

2. *Sur les axiomatiques du calcul des probabilités et leurs relations avec les expériences*, by W. Feller. The first part of this paper contains a general discussion of the foundations of the theory of probability. The second part is devoted to an exposition and critique of E. Tornier's theory of probability. The exposition is much clearer than that found in Tornier's book (*Wahrscheinlichkeitsrechnung und algemeine Integrationstheorie*). Feller criticizes Tornier's objections to the complete additivity in Kolmogoroff's system. He notes Tornier's failure to recognize that his system is a special case of that of Kolmogoroff and that his use of the continuity axiom is precisely that of Kolmogoroff. To these criticisms I should like to add the following.

In order to avoid the difficulties of the Regellosigkeitsprinzip of von Mises, Tornier constructs a theory which is oversimplified to such an extent that it presents a much less accurate picture of the physical universe. This sacrifice is unnecessary. See for example the paper by Wald or the modified systems described by Fréchet in this volume.

3. *Exposé et discussion de quelques recherches récentes sur les fondements du calcul des probabilités*, by M. Fréchet. The author starts with the classical concept of probability of Laplace and Poincaré. He recalls the familiar objections to this point of view and indicates how these objections might be overcome. Next he presents the theory of von Mises and the objections which have been raised against that theory. He also adds objections of his own. Then he describes the modifications of the von Mises theory by Popper, Reichenbach, Copeland, Wald, and Ville. He argues that although these modifications avoid the serious logical difficulties of the von Mises theory, they are open to a new objection, that is, that they are less natural and do not possess the strong intuitional appeal of the original theory. The author then turns to a point of view which he finds more acceptable and which has been described by Neyman as the modernized classical theory. This point of view is based on the postulates of Kolomogoroff and the modern theory of measure. Whether or not one agrees with the author he cannot fail to recognize that this article contains an excellent exposition of the various points of view and a clear presentation of the difficulties which attend them.

4. *Quelques remarques sur les fondements du calcul des probabilités*, by R. de Misès. The author gives a brief exposition of his own frequency theory of probability and points out the analogy between it and the theory of mechanics. He accepts the refinements contributed by Wald, Copeland, and other authors but states that these refinements do not alter essentially the fundamental character of his theory. As to Fréchet's contention that these refinements tend to rob his theory of its intuitive appeal, he points out that this is the case with any mathematical concept but that such refinements are necessary for a firmer foundation. He remarks that the modernized classical theory of probability is insufficient for the solution of applied problems and that without the frequency theory the modern methods such as chains of probabilities, convergence in probability, and arbitrary functions become merely problems in algebra and point set theory.

5. *Fréquence et probabilité*, by J. F. Steffensen. Steffensen upholds the frequency theory of probability as being more natural and more easily applicable in problems of statistics. He bases his discussion of

this theory on two hypotheses: first, that every possible sequence of successes and failures should be taken into consideration and second that if two fortuitous sequences are continued sufficiently far they will eventually differ. From these hypotheses he concludes that no sequence determined a priori by mathematical law can occur. He states that the probability approaches 1 that the relative frequency will differ from the probability by an arbitrarily small amount and gives a new proof of this fact. He then shows that the additive and multiplicative laws of probability can be similarly treated.

6. *Die Widerspruchsfreiheit des Kollektivbegriffes*, by A. Wald. A Kollektiv is an infinite sequence of elements taken from a given label space. Probabilities are associated with subsets of this label space and these probabilities are defined as limits of success ratios with respect to the Kollektiv. These limits must be invariant under the operation of selection. Thus far Wald agrees with von Mises. However instead of the absolute Kollektiv of von Mises, Wald's Kollektiv is relative to a given family of subsets of the label space and to a given system of selections. That is, probabilities are required to exist only for subsets belonging to the family, and invariance is required only with respect to selections of the given system. He shows that if the family of sets constitutes a denumerable field including the label space and if the system of selections is also denumerable, then there corresponds a continuum of Kollektivs. Thus the system can contain all selections which can be defined within a given symbolic logic. He shows further that if the denumerable field and the denumerable system are constructively defined, then there exist Kollektivs which can be constructively defined. The remainder of the paper is devoted to showing the relation of his system to those of Reichenbach, Popper, and Copeland and to answering objections raised against the frequency theory.

Volume III. *Les Sommes et les Fonctions de Variables Aléatoires*. 1938. 68 pp.

1. *Sur un nouveau théorème-limit de la théorie des probabilités*, by H. Cramér. The author considers the probability that $Z_1+Z_2+\cdots+Z_n\leqq x\cdot\sigma\cdot(n)^{1/2}$ where $Z_1, Z_2, \cdots$ are independent fortuitous variables having a common distribution function and such that the expected value of each is 0 and the expected value of the square of each is σ^2. It is well known that as n becomes infinite this probability tends to the probability integral if x is constant with respect to n. However the author is interested in this limit when x is allowed to become infinite with n. By means of a certain transformation using

characteristic (or moment generating) functions this problem can be studied with the aid of Liapounoff's theorem. He considers the case in which x is of order $n^{1/2}/\log n$ and the case in which x is of order $n^{1/2}$. He obtains theorems of Khintchine, Lévy, and Smirnoff as special cases of his results. In the latter part of the paper he applies his method to the study of homogeneous stochastic processes.

2. *L'arithmétique des lois de probabilités et les produits finis de lois de Poisson*, by P. Lévy. This paper deals with the field consisting of all probability laws. A probability law can be defined either by means of a distribution function or by means of the corresponding characteristic function. If two independent fortuitous variables have respectively the probability laws L and L', then the sum of the two variables has the law denoted by the product LL'. The characteristic function corresponding to the law LL' is the product of the characteristic functions of the laws L and L'. The set of all probability laws constitutes a field in which a unity is a law corresponding to a fortuitous variable which can take on only a single value. This memoir is a study of the decompositions of the elements of the field into factors.

3. *Généralisation des théorèmes de limite classiques*, by R. de Misès. The author studies the asymptotic distributions of certain functions of n fortuitous variables when n is allowed to become infinite. As an aid in this study he considers functionals of the corresponding distributions. For example the moments of a given distribution are linear functionals of the corresponding distribution function. These functionals are defined over convex sets of distribution functions and derivatives are defined at points of these sets in such a manner as to permit Taylor's developments of the functionals.

Volume IV. *Le Principe Ergodique et les Probabilités en Chaîne*. 1938. 66 pp.

1. *Statistische Probleme und Ergebnisse in der klassischen Mechanik*, by E. Hopf. The equations of motion of a mechanical system in general define a one-parameter group of measure preserving transformations T_t of the phase space Ω into itself (the parameter t being the time). A transformation is said to be metrically transitive provided any subset of Ω which is transformed into itself has either the measure 0 or else the same measure as Ω. The condition of metric transitivity is necessary and sufficient that the limit average time that a point spends in a subset A of Ω is equal to the measure of A divided by the measure of Ω (Birkhoff's ergodic theorem). If the system is metrically transitive but fails to possess a characteristic frequency,

then every subset of Ω becomes uniformly distributed over Ω as the time is indefinitely increased (mixture theorem). The rest of the paper is devoted to the discussion of the weaker condition, regional transitivity.

2. *Les fluctuations* (*changement aléatoires du nombre de points ou d'objets dans un compartiment*), by B. Hostinsky. The author considers three problems. The first concerns the distribution of N objects into n compartments. He computes the probability that x objects will be placed in a given compartment assuming that $1/n$ is the probability that a given object will be placed in that compartment. The limiting case in which N and n become infinite gives the Poisson law. The second problem concerns two urns with r balls in each. There are r white and r black balls, but otherwise the compositions of the urns are arbitrary. From each urn a ball is drawn at random and then placed in the urn from which the other was drawn. The author considers the limiting probability of a given composition as the number of such exchanges becomes infinite. This problem is analogous to the mixture of a colored liquid and a non-colored liquid in two containers which are connected. The third problem concerns two urns containing a total of three balls of the same color. Probabilities are given for the transference of a ball from one urn to the other and for failure to make such transference. These probabilities depend on the compositions of the urns. The limiting probabilities for various compositions are again computed. This problem is analogous to the exchange of compressible fluids or gases between two connected containers.

3. *Théorie générale des chaînes à liaisons complètes*, by O. Onicescu. The author considers a matrix composed of mutually exclusive fortuitous events (or states) $a_1, a_2, \cdots, a_m$ and the probabilities $x_1, x_2, \cdots, x_m$ corresponding to these events. Such a matrix can be transformed into a new matrix with new probabilities $y_i = f_{ik}(x_1, x_2, \cdots, x_m)$ for the given events, k being determined by the event a_k which does occur. Such a system of transformations constitutes a chain. If the matrix f_{ik} is independent of the probabilities, the transformation is completely fortuitous and the system constitutes a chain with complete liaisons. A chain is normal if the matrix f_{ik} possesses the ergodic property enjoyed by the Markoff chains. Such a matrix transforms the region $0 \leqq x_i \leqq 1$ into a closed region totally interior to this, and the limiting region as the number of transformations becomes infinite is a single point. The author considers also the continuous case in which the number of states is infinite. He cites as an example the equation of the flow of heat.

4. *Quelques problèmes nouveaux de la théorie des chaînes de Markoff*, by V. Romanovsky. The author considers a simple Markoff chain and a second chain such that there is a definite conditional probability of a given state on a given trial for every possible state of the corresponding trial of the first chain. The second chain is in general multiple and nonhomogeneous. A method is given for computing moments and the limiting correlation coefficient with respect to the chains. The author next studies cyclic chains. These chains may contain several cycles which occur fortuitously. A method of matrices is used to simplify the study of these chains. The latter part of the paper is devoted to methods of estimating by statistical means the law underlying a Markoff chain.

Volume V. *Les Fonctions Aléatoires*. 1938. 73 pp.

1. *Équations différentielles stochastiques*, by S. Bernstein. This paper is concerned with a difference equation of the form $\Delta y = A\Delta t + f(\Delta t)^{1/2}$ where A and f depend on y, t, Δt, and a fortuitous variable α_t and where the expected value of f is 0. Sufficient conditions are given for the existence of a limiting distribution function for y as the differences Δt approach 0. Bernstein shows that these conditions cannot be materially weakened. If $f \equiv 0$ and A is not a function of α_t, the problem reduces to the solution of the differential equation $dy/dt = A(y, t)$. Moreover if $(\Delta t)^{1/2}$ is replaced by Δt in the second term of the right-hand member of the difference equation and A is independent of α_t then y satisfies the equation $dy/dt = A(y, t)$ and the solution does not possess a fortuitous character. The method of this paper can be extended to systems of difference equations of the above form.

2. *Sur les fonctions aléatoires presque périodiques et sur la décomposition des fonctions aléatoires stationnaires en composantes*, by E. Slutsky. A fortuitous function y_t is assumed to be stationary, normalized, stochastically continuous, and subject to certain other minor conditions. The correlation $\rho(\tau)$ between y_t and $y_{t+\tau}$ can be decomposed into a constant plus a component with a discrete spectrum plus a component with a continuous spectrum. The portion with the discrete spectrum is an almost periodic function developable in a Fourier series. The function y_t admits of a decomposition into three non-correlated components the first of which is the mean value of y_t the second of which consists of a Fourier series closely related to the corresponding series for $\rho(\tau)$, and the third of which has a spectrum continuous in the mean. If $\rho(\tau)$ is almost periodic, then the probability is 1 that the Fourier series for y_t converges to y_t for almost every t.

3. *La théorie et les applications des fonctions indépendantes au sens stochastique*, by H. Steinhaus. If $f(t)$ is defined in the interval 0, 1, then the corresponding distribution function $F(x)$ is defined as the measure of the set of points on which $f(t) < x$ and is interpreted as the probability of this latter inequality. By suitable modifications this definition can be extended to any finite interval and also to the infinite interval. Independence is defined as independence in the probability sense. The functions $f(t)$, $g(t)$ defining the space filling curve of Peano furnish an example of independent functions. Such independent functions are an aid to calculation in many probability problems. The author applies this method to the probability limit theorem, the Maxwell distribution law, and to Brownian movements.

Volume VI. *Conceptions Diverses*. 1938. 57 pp.

1. *Sur la conception d'équivalence partielle*, by B. de Finetti. Finetti bases his theory of probability on the assumption that the laws of total and composite probabilities are valid and on a concept which he calls the equivalence of events. Events are said to be equivalent provided the probability that n of them will occur depends only on n and not on the particular set of n events chosen. He derives a formula (not the classical and highly dubious application of Bayes' principle) by means of which one can estimate the probability that one of the equivalent events will occur on the basis of an observed success ratio. Except for a very special case the formula shows that this probability must be close to the success ratio if the number of trials is large and that the events must be independent. The formula can be extended to groups of events where events in given groups are equivalent. This paper also contains a philosophical discussion concerning the relations between probability, causality, and induction.

2. *Sur la loi des grands nombres dans l'espace fonctionnel*, by V. Glivenko. Suppose that a fortuitous event has a continuous probability density defined in a closed interval. A histogram represents the observed frequency distribution in n trials of the event. The author shows that the probability is 1 that the maximum difference between the histogram and the probability density approaches 0 as the number of trials becomes infinite. This result is based on certain theorems which the author proves concerning moments of fortuitous functions. In particular he proves a generalization of the Bienaymé-Tchebycheff inequality and the strong law of large numbers in function space. The article though brief is complete and little previous knowledge of the theory of probability is assumed.

3. *Les fonctions aléatoires (lois de répartition dans un espace fonc-*

tionnel) et leurs applications, by A. Kolmogoroff. This paper was not completed at the time these volumes were published. It will appear in the collection *Exposés d'Analyse Générale* published under the direction of M. Fréchet.

4. *L'estimation statistique, traitée comme un problème classique de probabilité*, by J. Neyman. Let $X_1, X_2, \cdots, X_n$ be a system of fortuitous variables depending on the unknown parameters $\theta_1, \theta_2, \cdots, \theta_s$. An experiment determines the values of the fortuitous variables, that is, a point of the euclidean space R_n. Let a set function of points be defined such that to every point of R_n is assigned a set of points of the real axis. If this function is properly determined, then when $X_1, X_2, \cdots, X_n$ is substituted for the point of R_n there will be determined a probability that a given point θ_1 of the real axis will be included in such a set, θ_1 being one of the parameters. Let the function be further restricted so that this probability is a preassigned number α (the coefficient of confidence) for all possible values of the parameters. It follows that when an experiment determines a point of R_n the probability is α that the parameter θ_1 will be found in the corresponding set. Suppose that the number of parameters is 1 and that the above function is further restricted so that the probability α is maximized (maximum likelihood). The sets (functions of points of R_n) are then intervals (confidence intervals) and are uniquely determined.

Volume VII. *La Statistique Mathématique*. 1938. 57 pp.

1. *Certain coefficients of regression or trend associated with largest likelihood*, by E. Dodd. A pair of fortuitous variables x, y is assumed to be subject to a probability distribution which is known except for the values of certain parameters. A set of observations is made determining n values of each variable. The parameters are then determined by maximizing the likelihood of obtaining the values which were actually obtained. The author considers two examples and in each case the parameters are determined as certain means closely related to regression coefficients. The author then considers a general concept of mean and a corresponding theorem concerning maximum likelihood. However the conditions of this theorem are such as to permit an arbitrary assignment of value to the unknown parameter.

2. *Critique de la corrélation au point de vue des probabilités*, by C. Jordan. The author considers the correlation between two fortuitous variables u and v. A measure called the intensity of correlation is defined in terms of the discrepancies between certain conditional probabilities and corresponding unconditional probabilities. This

measure is equal to 0 if and only if the variables are independent and it attains its maximum 1 if and only if the variables are completely dependent. Although the intensity of correlation is never negative and hence cannot distinguish between positive and negative correlation, the author gives a simple independent method of accomplishing this distinction. The intensity of correlation is closely related to two measures given respectively by Pearson and Tschuprow. Jordan also discusses the conventional correlation coefficient, two additional measures due to Pearson, and Pearson's correlation ratios. The intensity of correlation is the only one of these measures which possesses all of the above mentioned desirable properties.

3. *Sur la loi de Poisson, la loi de Charlier et les équations linéaires aux différences finies du premier ordre à coefficients constants*, by N. Obrechkoff. The first portion of this paper concerns a system of functions consisting of the Poisson function $(a^x/x!)e^{-a}$ and its differences. Any function of the system is the product of the Poisson function with the appropriate Charlier polynomial. The author gives sufficient conditions for the development of a given function in a series formed from this system. The second part of the paper deals with a pair of dependent fortuitous variables such that each has a Poisson distribution when the other is fixed. This condition determines the formulas for the distribution of the two variables and the unrestricted distribution of each of the variables. The last part of the paper concerns a certain type of linear nonhomogeneous difference equation. The author gives a formula for the asymptotic behavior of the solution.

A. H. COPELAND

Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung. By Karl Stumpff. Berlin, Springer, 1939. 172 pp.

This book is a sequel to the author's volume *Grundlagen und Methoden der Periodenforschung* (Berlin, Springer, 1937). It contains extensive tables for the carrying out of harmonic analysis of empirical curves based upon interpolation by means of trigonometric polynomials of proper degree through equally spaced ordinates. Tables are given for various numbers p of equal divisions. Tables are given for the proper sines and cosines by which to multiply the p ordinates in order to obtain the proper Fourier coefficients. These tables extend as far as $p=40$. They also include the best laborsaving arrangement for the ordinates. These tables are followed by the first thousand multiples of cosines and sines of various angles which occur for $p=8$, 12, 16, and 24, as well as the first hundred multiples of the cosines and sines of all angles of integral degrees in the first quadrant. These