TRANSFORMATION THEORY OF INTEGRABLE DOUBLE-SERIES OF LINEAL ELEMENTS[1]

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1. Introduction. We shall begin by giving some of the concepts in the geometry of lineal elements of three-space. By a *lineal element*, we mean simply a point and a direction (of a straight line) through that point. A lineal element may be defined by the five numbers (x, y, z, p, q) where (x, y, z) are the cartesian coordinates of the point and $(1, p, q)$ are the direction numbers of the direction of the element. We shall call the ∞^5 lineal elements of space a *plenum*.

A set of ∞^1 lineal elements of space is termed a *series*. A series may usually be pictured as the configuration obtained by attaching to each point of a curve a single direction (usually not the tangent direction). However, there is a degenerate type of series called the *point-union*, or *conical-union*. This consists of ∞^1 lineal elements through a fixed point. Thus the lineal elements of a point-union form a cone with the fixed point as vertex. A series may be given, in general, by the four equations $y=y(x)$, $z=z(x)$, $p=p(x)$, $q=q(x)$ where y, z, p, q are arbitrary functions of x only.

A *union* is a series which either consists of a curve together with the tangent directions of the curve, or as a special case, is a point-union. *The necessary and sufficient conditions that the series $y=y(x)$, $z=z(x)$, $p=p(x)$, $q=q(x)$ be a union are $dy/dx=p$, $dz/dx=q$.*

A collection of ∞^2 lineal elements of space is called a *double-series*. A double-series is the gemetric object obtained by attaching to each point of a surface a single direction (usually not a tangent direction). But there is a degenerate type of double-series, called the *point-union double-series*. This consists of ∞^1 point-unions. That is, if at each point of a curve, we construct ∞^1 lineal elements (a cone of lineal elements), the resulting configuration is a point-union double-series. There is another type of point-union double-series which is called a *star* or *bundle*. In this case, all the point-unions are concurrent so that a star or bundle consists of all the ∞^2 lineal elements through a given point. A double-series may be given by the three equations $z=z(x, y)$, $p=p(x, y)$, $q=q(x, y)$, where z, p, q are arbitrary functions of (x, y) only.

If we can find ∞^1 unions whose lineal elements coincide exactly with the ∞^2 lineal elements of a given double-series, then the double-series is said to be an *integrable double-series*. This means that an in-

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tegrable double-series is either the configuration obtained by constructing at each point of a surface a single tangent direction to the surface, or as a special case, a point-union double-series. (The unions of a point-union double-series are the ∞^1 point-unions which determine the double-series.) *The necessary and sufficient condition that the double-series $z=z(x, y)$, $p=p(x, y)$, $q=q(x, y)$ be an integrable double-series is that the functions z, p, q of (x, y) satisfy the partial differential equation of first order $q=z_x+pz_y$.*

A set of ∞^3 lineal elements of space is said to be a *field*. A field is the geometric object obtained by constructing at each point of space a single direction. However, there is a degenerate type of field, called the *point-union field*. This consists of ∞^2 point-unions. If at each point of a surface we construct ∞^1 lineal elements (a cone of lineal elements), the resulting object is a point-union field. Another type of point-union field is the *bundle-field*. This consists of ∞^1 stars or bundles. That is, if at each point of a curve, we construct all the ∞^2 lineal elements through the point, the result is a bundle-field. Of course, each of the ∞^1 bundles of a bundle-field contains ∞^1 concurrent point-unions so that a bundle-field consists of ∞^2 point-unions. A field may be given by the two equations $p=p(x, y, z)$, $q=q(x, y, z)$ where p and q are arbitrary functions of (x, y, z) only.

A field corresponds to two ordinary differential equations of the first order in the unknowns y and z considered as functions of x only. In other words, a field is determined by the two ordinary equations of first order $dy/dx=p(x, y, z)$, $dz/dx=q(x, y, z)$, where p and q are arbitrary functions of (x, y, z) only. By this, we find that *every field is an integrable field*. That is, by integrating our equations, we can always find ∞^2 unions whose lineal elements coincide exactly with the ∞^3 lineal elements of the given field. (Thus the unions of a point-union field are the ∞^2 point-unions which determine the field.)

A collection of ∞^4 lineal elements of space is termed a *conical-field*. A conical-field is the configuration obtained by constructing at each point of space ∞^1 lineal elements (a cone of lineal elements). But there is a degenerate type of conical-field called the *point-union conical-field*. This is the result of constructing at each point of a surface the bundle of ∞^2 lineal elements through that point. Thus a point-union conical-field consists of ∞^2 bundles. Since a bundle consists of ∞^1 concurrent point-unions, we find that a point-union conical-field consists of ∞^3 point-unions. A conical-field may be given by the equation $q=q(x, y, z, p)$ where q is an arbitrary function of (x, y, z, p) only.

A conical-field corresponds to a single Monge equation of the first order in the unknowns y and z considered as functions of x only.

In other words, a conical-field is determined by the Monge equation of first order $dz/dx = q(x, y, z, dy/dx)$. *Thus every conical-field is an in-integrable conical-field.* That is, by setting y equal to an arbitrary function of x and upon integrating the resulting ordinary differential equation of the first order in the unknown z, we can find ∞^∞ unions, whose lineal elements coincide exactly with the ∞^4 lineal elements of the given conical-field. Also every conical-field contains ∞^3 point-unions. (Thus the unions of a point-union conical-field are the ∞^∞ curves of the basic surface of the point-union conical-field and also the ∞^3 point-unions of the ∞^2 bundles which determine the conical-field.)

A transformation between the lineal elements of space is called a *lineal element transformation.* Any lineal element transformation is given by the equations

$$\begin{gathered} X = X(x, y, z, p, q), \quad Y = Y(x, y, z, p, q), \quad Z = Z(x, y, z, p, q), \\ P = P(x, y, z, p, q), \quad Q = Q(x, y, z, p, q), \end{gathered} \tag{1}$$

where X, Y, Z, P, Q are arbitrary functions of (x, y, z, p, q) only such that the jacobian is not identically zero:

$$J = \begin{vmatrix} X_q & X_p & X_z & X_y & X_x \\ Y_q & Y_p & Y_z & Y_y & Y_x \\ Z_q & Z_p & Z_z & Z_y & Z_x \\ P_q & P_p & P_z & P_y & P_x \\ Q_q & Q_p & Q_z & Q_y & Q_x \end{vmatrix} \not\equiv 0. \tag{2}$$

A lineal element transformation converts every series into a series, but it does not in general convert every union into a union. In the paper by Kasner, *General transformation theory of differential elements*, American Journal of Mathematics, vol. 32 (1904), pp. 391–401, the following fundamental theorem is proved.

All lineal element transformations of spaces may be classified with respect to the number of unions preserved into three distinct types:

TYPE 1. *The infinite group of extended point transformations. Any lineal element transformation which converts every union into a union must be an extended point transformation of the form*

$$\begin{gathered} X = X(x, y, z), \qquad Y = Y(x, y, z), \qquad Z = Z(x, y, z), \\ P = \frac{Y_x + pY_y + qY_z}{X_x + pX_y + qX_z}, \qquad Q = \frac{Z_x + pZ_y + qZ_z}{X_x + pX_y + qX_z}, \end{gathered} \tag{3}$$

where X, Y, Z are arbitrary functions of (x, y, z) only.

TYPE 2. *The set of lineal element transformations, not extended point transformations, fulfilling the conditions*

$$(4)\qquad \frac{Z_q - QX_q}{Y_q - PX_q} = \frac{Z_p - QX_p}{Y_p - PX_p} = \frac{Z_x + pZ_y + qZ_z - Q(X_x + pX_y + qX_z)}{Y_x + pY_y + qY_z - P(X_x + pX_y + qX_z)} .$$

Any lineal element transformation of this set converts ∞^∞ *unions into unions, but not every union into a union. That is, the family of unions preserved involves an arbitrary function. This family is defined by a Monge equation of the second order of the form*

$$(5)\qquad E\frac{d^2y}{dx^2} + F\frac{d^2z}{dx^2} + G = 0,$$

where E, F, G *are arbitrary functions of* $(x, y, z, dy/dx, dz/dx)$ *only. Any Monge equation of the above form is characterized by the possession of the Meusnier property.* (See Kasner, *The inverse of Meusnier's theorem*, this Bulletin, vol. 14 (1908), pp. 461–465.)

TYPE 3. *The set of lineal element transformations not fulfilling the conditions* (4). *Any lineal element transformation of this set converts exactly* ∞^4 *unions into unions.* (This is the most general case.)

A field is said to be a *normal field* if we can construct for the doubly infinite family of integral unions of the field ∞^1 orthogonal surfaces. Of course not every field is a normal field. Under an arbitrary lineal element transformation, every field is converted into a field, but every normal field is not converted in general into a normal field. In the paper by Kasner, *Lineal element transformations of space for which normal congruences of curves are converted into normal congruences*, Duke Mathematical Journal, vol. 5 (1939), pp. 72–83, the following result is derived.

The infinite group of lineal element transformations which convert every normal field into a normal field is isomorphic with the Lie group of contact transformations of surface elements.

Next we come to the *new problem* of the present paper. This will complete the list of fundamental problems in the theory of transformations of differential elements.

A lineal element transformation converts every double-series into a double series, but it does not in general transform every integrable double-series into an integrable double-series. The problem of this paper is *to find the group of lineal element transformations which carry every integrable double-series into an integrable double-series.* Our result is that *our group of lineal element transformations is the group of extended point transformations.* Thus the solution of our problem gives us a new characteristic property of the group of extended point transformations. (See the paper by Kasner and De Cicco, *Curvature element transformations which preserve integrable fields*, Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 104–111, where an analogous characterization of the group of extended contact transformations of lineal elements of the plane is given.)

We note that we do *not* assume that the individual unions in the family of ∞^1 unions are converted into unions. But from our proof it *does* result that if *every* integrable double-series becomes such a double-series, then the individual unions are actually converted into individual unions, and *therefore*, the result is our extended point transformation.

2. Integrable double-series into integrable double-series. In this section, we shall state and prove our result.

FUNDAMENTAL THEOREM. *The group of lineal element transformations which convert every integrable double-series into an integrable double-series is the group of extended point transformations.*

The sufficiency of our theorem is obvious. The remainder of the paper is concerned with the proof of the necessity of the theorem.

Let the lineal element transformation T as given by equations (1) carry every integrable double-series into an integrable double-series. Then there are ∞^∞ integrable double-series of the form $z=z(x, y)$, $p=p(x, y)$, $q=q(x, y)$ which are carried into the integrable double-series of the form $Z=Z(X, Y)$, $P=P(X, Y)$, $Q=Q(X, Y)$.

Therefore for these integrable double-series, the condition $q=z_x+pz_y$ must be transformed by T into the condition

$$Q = Z_x + PZ_y. \tag{6}$$

The equation (6) may be written, where $z_x=q-pz_y$, in the form

$$\begin{aligned} &Q\begin{vmatrix} X_x+X_zz_x+X_pp_x+X_qq_x & X_y+X_zz_y+X_pp_y+X_qq_y \\ Y_x+Y_zz_x+Y_pp_x+Y_qq_x & Y_y+Y_zz_y+Y_pp_y+Y_qq_y \end{vmatrix} \\ &= \begin{vmatrix} Z_x+Z_zz_x+Z_pp_x+Z_qq_x & Z_y+Z_zz_y+Z_pp_y+Z_qq_y \\ Y_x+Y_zz_x+Y_pp_x+Y_qq_x & Y_y+Y_zz_y+Y_pp_y+Y_qq_y \end{vmatrix} \\ &+ P\begin{vmatrix} X_x+X_zz_x+X_pp_x+X_qq_x & X_y+X_zz_y+X_pp_y+X_qq_y \\ Z_x+Z_zz_x+Z_pp_x+Z_qq_x & Z_y+Z_zz_y+Z_pp_y+Z_qq_y \end{vmatrix}. \end{aligned} \tag{7}$$

The equation (7) must be an identity in z_y, p_x, p_y, q_x, q_y, after we eliminate z_x. Upon setting the coefficients equal to zero and simplifying, we obtain the *nine* partial differential equations of first order

$$
(8)\quad
\begin{gathered}
\begin{vmatrix} Z_q - QX_q & Z_p - QX_p \\ Y_q - PX_q & Y_p - PX_p \end{vmatrix} = 0, \quad
\begin{vmatrix} Z_p - QX_p & Z_z - QX_z \\ Y_p - PX_p & Y_z - PX_z \end{vmatrix} = 0, \\
\begin{vmatrix} Z_p - QX_p & Z_y - QX_y \\ Y_p - PX_p & Y_y - PX_y \end{vmatrix} = 0, \quad
\begin{vmatrix} Z_q - QX_q & Z_z - QX_z \\ Y_q - PX_q & Y_z - PX_z \end{vmatrix} = 0, \\
\begin{vmatrix} Z_q - QX_q & Z_y - QX_y \\ Y_q - PX_q & Y_y - PX_y \end{vmatrix} = 0, \quad
\begin{vmatrix} Z_q - QX_q & Z_x - QX_x \\ Y_q - PX_q & Y_x - PX_x \end{vmatrix} = 0, \\
\begin{vmatrix} Z_p - QX_p & Z_x - QX_x \\ Y_p - PX_p & Y_x - PX_x \end{vmatrix} = 0, \\
\begin{vmatrix} Z_x + pZ_y - Q(X_x + pX_y) & Z_z - QX_z \\ Y_x + pY_y - P(X_x + pX_y) & Y_z - PX_z \end{vmatrix} = 0, \\
\begin{vmatrix} Z_x + qZ_z - Q(X_x + qX_z) & Z_y - QX_y \\ Y_x + qY_z - P(X_x + qX_z) & Y_y - PX_y \end{vmatrix} = 0.
\end{gathered}
$$

In the rest of the paper, we shall concern ourselves with the complete solution of these equations. We shall show that the only possible solution for the five functions X, Y, Z, P, Q with nonvanishing jacobian is furnished by the equations (3), and therefore our transformation is an extended point transformation.

We shall prove that

$$(9)\ Z_q - QX_q = 0,\ Z_p - QX_p = 0,\ Y_q - PX_q = 0,\ Y_p - PX_p = 0.$$

Let us suppose that this is not the case. That is, let us assume that at least one of the quantities of the left-hand sides of the preceding equations is different from zero. From the first seven of the equations (8), we obtain

$$
(10)\quad
\begin{aligned}
\frac{Z_q - QX_q}{Y_q - PX_q} = \frac{Z_p - QX_p}{Y_p - PX_p} &= \frac{Z_z - QX_z}{Y_z - PX_z} \\
&= \frac{Z_y - QX_y}{Y_y - PX_y} = \frac{Z_x - QX_x}{Y_x - PX_x}.
\end{aligned}
$$

But these equations make the jacobian (2) of our transformation zero. This contradiction establishes the equations (9).

Since the equations (9) are satisfied, we find that the first seven of equations (8) are zero. Therefore the equations (8) by means of equations (9) reduce to the *two* partial differential equations of first order

$$(11)\qquad \begin{vmatrix} Z_x + pZ_y - Q(X_x + pX_y) & Z_z - QX_z \\ Y_x + pY_y - P(X_x + pX_y) & Y_z - PX_z \end{vmatrix} = 0, \qquad \begin{vmatrix} Z_x + qZ_z - Q(X_x + qX_z) & Z_y - QX_y \\ Y_x + qY_z - P(X_x + qX_z) & Y_y - PX_y \end{vmatrix} = 0.$$

We shall prove that

$$(12)\qquad \begin{aligned} Z_x + pZ_y + qZ_z - Q(X_x + pX_y + qX_z) &= 0, \\ Y_x + pY_y + qY_z - P(X_x + pX_y + qX_z) &= 0. \end{aligned}$$

Let us suppose that this is not the case. That is, let us assume that at least one of the quantities of the left-hand sides of the preceding equations is not zero. From equations (11), we find after simplification

$$(13)\qquad \frac{Z_z - QX_z}{Y_z - PX_z} = \frac{Z_y - QX_y}{Y_y - PX_y} = \frac{Z_x - QX_x}{Y_x - PX_x}\,.$$

These equations together with the equations (9) make the jacobian of our transformation zero. This contradiction establishes the equations (12).

By means of (9) and (12), we have found that the complete solution of equations (8) is given by the equations

$$(14)\qquad \begin{gathered} Z_x + pZ_y + qZ_z = Q(X_x + pX_y + qX_z), \\ Y_x + pY_y + qY_z = P(X_x + pX_y + qX_z), \\ Z_p = QX_p, \qquad Z_q = QX_q, \qquad Y_p = PX_p, \qquad Y_q = PX_q. \end{gathered}$$

We shall show that $X_q = X_p = 0$. Let us suppose that at least one of these quantities is not zero. Upon taking the partial derivative with respect to q of the third and the fifth of the preceding equations, we obtain

$$(15)\qquad Z_{pq} = QX_{pq} + Q_qX_p, \qquad Y_{pq} = PX_{pq} + P_qX_p.$$

Upon taking the partial derivative with respect to p of the fourth and the sixth of the equations (14), we find

$$(16)\qquad Z_{pq} = QX_{pq} + Q_pX_q, \qquad Y_{pq} = PX_{pq} + P_pX_q.$$

Subtracting the corresponding equations of (15) and (16) and comparing the results with the last four of equations (14), we obtain

$$(17)\qquad Q_q/Q_p = P_q/P_p = Z_q/Z_p = Y_q/Y_p = X_q/X_p.$$

But these equations make the jacobian (2) of our transformation zero. This contradiction proves our assertion that $X_p = X_q = 0$.

Since $X_q = X_p = 0$, we find from equations (14) that $Z_q = Z_p = Y_q = Y_p = X_q = X_p = 0$. Therefore the equations (14) become the equations (3). This proves that our lineal element transformation is an extended point transformation. The proof of our fundamental theorem is complete.

3. **General lineal element transformations.** A lineal element transformation, not an extended point transformation, will convert *some* particular integrable double-series into integrable double-series. This subject will be considered elsewhere.

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