so that $\sum_{m=}^{\infty} |A_m(f, 0)| = \infty$. It remains to show that $f(x) \subset L$ which is easily seen since

$$\int_{-\pi}^{\pi} | f(x) | \, dx = \sum_{i=0}^{\infty} 2^{-i} \int_{-\pi}^{\pi} | f_{n_i}(x) | \, dx$$

$$\leqq \sum_{i=0}^{\infty} 2^{-i} 2(n+1) \frac{\pi}{3(n+1)} < \infty .$$

We notice that, since this function vanishes in the neighborhood of the origin, it coincides with a function having an absolutely summable Fourier series in the neighborhood of the origin, and therefore absolute summability $C(1)$ is not a local property.

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COMPLETE REDUCIBILITY OF FORMS[1]

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1. **Introduction.** We shall say that F is a form in *r essential variables* with respect to a field K if F cannot be brought by means of a nonsingular linear transformation in the field K to a form with less variables. Let F be a form of degree p written as $a_{ij\cdots k}x_ix_j \cdots x_k$, $(i, j, \cdots, k=1, 2, \cdots, n)$. We arrange the coefficients of F in a matrix A whose n^{p-1} columns are of the form

$$\left\| \begin{matrix} a_{1j\cdots k} \\ a_{2j\cdots k} \\ \cdot \\ \cdot \\ \cdot \\ a_{nj\cdots k} \end{matrix} \right\| .$$

The index i is associated with the rows of A and the $p-1$ indices $j, \cdots, k$ are associated with the columns of A. We assume that the coefficients in F are so chosen that A is *symmetric* in the sense that the value of an element $a_{ij\cdots k}$ is unchanged under permutation of the subscripts. It can be shown[2] that F is *a form in r essential variables if and only if the rank of A is r.*

A form F is said to be *completely reducible* in a field K if F splits

[1] Presented to the Society, April 7, 1939.

[2] Oldenburger, *Composition and rank of n-way matrices and multilinear forms,* Annals of Mathematics, (2), vol. 35 (1934), pp. 622–653.

in K into a product of linear factors. Hočevar proved[3] that a form F with no repeated factors is completely reducible in the *complex field* if and only if F divides each third order minor of its Hessian. It is obvious that this result of Hočevar is not valid for each field of numbers. A form F of degree p is said to be *nonsingular with respect to K* if F can be written as a linear combination of pth powers of linearly independent linear forms with coefficients in K. Elsewhere the author proved[4] that the Hessian of a cubic form nonsingular with respect to K factors in K into linearly independent factors. For a field K with characteristic different from 2, 3, and element $a \neq 0$, the product $ax_1x_2 \cdots x_n$ in n independent variables $x_1, x_2, \cdots, x_n$ is the Hessian of the nonsingular cubic $C(a)$ where $6C(a) = ax_1^3 + x_2^3 + \cdots + x_n^3$. We let $L_i = b_{ij}y_j$, $(i, j = 1, 2, \cdots, n)$, denote an arbitrary set of n linear forms linearly independent with respect to K. We write Δ for the determinant of the matrix (b_{ij}). Applying the nonsingular linear transformation $x_1 = L_1, x_2 = L_2, \cdots, x_n = L_n$ to $C(1/\Delta^2)$ we obtain a form whose Hessian is $L_1L_2 \cdots L_n$. Hence each product of linearly independent linear forms is the Hessian of a nonsingular cubic form. We have proved the theorem which follows.

THEOREM 1. *Let K be a field with characteristic not 2 or 3. A form F of degree n in n essential variables is completely reducible in K if and only if F can be written as the Hessian of a cubic form nonsingular with respect to K.*

If F of Theorem 1 is completely reducible and F is the Hessian of a nonsingular cubic form C, then $C = a_iL_i^3$, $(i = 1, 2, \cdots, n)$, and the linear forms $L_1, \cdots, L_n$ are the factors of F.

The utility of Theorem 1 is limited by the fact that the problem of representability of a form as the Hessian of a nonsingular cubic is unsolved. In the present paper we prove that a certain integer, called "minimal number," associated with a completely reducible form F of degree n is not greater than 2^{n-1}. From this property we obtain a *solution of the problem of complete reducibility of cubic forms* for a field K with characteristic not 2 or 3.

2. **Minimal numbers and representations.** Elsewhere[5] the author proved that each symmetric form F of degree p can be written for a

[3] Hočevar, *Sur les formes décomposables en facteurs linéaires*, Comptes Rendus de l'Académie des Sciences, vol. 138 (1904), pp. 745–747.

[4] Oldenburger, *Rational equivalence of a form to a sum of pth powers*, Transactions of this Society, vol. 44 (1938), pp. 219–249; in particular p. 233.

[5] Oldenburger, *Representation and equivalence of forms*, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 193–198.

field K of order p or more as a linear combination of pth powers of linear forms. Such a linear combination with ρ terms we call a ρ-representation of F with respect to K. A representation of F with respect to K with a minimum number of terms is called a *minimal representation* of F with respect to K. The number of terms in such a representation we term the *minimal number* of F with respect to K, and denote this number by $m(F)$.

THEOREM 2. *Let K be a field with characteristic*[6] *greater than n, and let F be a form of degree n completely reducible in K. Then $m(F) \leqq 2^{n-1}$.*

We write $\rho = 2^{n-1}$. Let $L_1, L_2, \cdots, L_\rho$ denote the different possible forms of the type $(x_1 \pm x_2 \pm x_3 \pm \cdots \pm x_n)$. Let $k_i = +1$ if L_i contains an even number of minus coefficients, and $k_i = -1$ if L_i contains an odd number of such coefficients. We consider the sum

$$\frac{1}{2^{n-1}}\left[\sum_{i=1}^{\rho} k_i L_i^n\right]. \tag{1}$$

Simple computation reveals that (1) is symmetric in the x's. We consider a product $\Pi = \pm x_1^a \cdots x_r^d$ of degree n with $r < n$ arising from the expansion of a term $k_i L_i^n$ in (1). Corresponding to the linear form L_i there is a unique form L_j, $(j \neq i)$, in (1) obtainable from L_i by changing the sign of x_n in L_i. Then $k_j = -k_i$. The product $P = x_1^a \cdots x_r^d$ arising from $k_j L_j^n$ has a coefficient the negative of that in Π. Thus the terms involving the product P, where these terms arise from $k_i L_i^p$ and $k_j L_j^p$, vanish. It follows that the coefficient of P in (1) is zero. It is obvious from the choice of the k_i that the coefficient of $x_1 \cdots x_n$ in (1) is $n!$, whence (1) is a ρ-representation of $n! x_1 \cdots x_n$. Since a completely reducible form F in n essential variables is equivalent to this product under nonsingular linear transformations in K, and *the minimal number is an invariant* of F, we have $m(F) \leqq 2^{n-1}$. It follows that if $F = L_1 L_2 \cdots L_n$ where $L_1, L_2, \cdots, L_n$ are linearly dedependent linear forms, $m(F) \leqq 2^{n-1}$.

3. **Complete reducibility of cubic forms.** In the present section we assume that the underlying field K is such that when two forms are equal to each other for all values of the variables in K, corresponding coefficients of these forms are equal. In the case of cubic forms this means that the characteristic of K is different from 2, 3. Evidently, *a completely reducible cubic form is a form in not more than 3 essential variables.* Since the minimal number of a binary cubic is not greater

[6] Restricting the characteristic of K to be greater than n is equivalent to assuming that the characteristic of K does not divide $n!$.

than 3, the theory of complete reducibility of binary forms may readily be supplied by the reader. In what follows we therefore consider cubic forms in 3 essential variables only.

THEOREM 3. *A cubic form F in 3 essential variables is completely reducible with respect to a field K if and only if*

(a) *The minimal number of F with respect to K is* 4.

(b) *If $\mu_i R_i^3$ is a minimal representation of F with respect to K, then roots $\sigma_i = (\mu_i/\mu_1)^{1/3}$ are in K for each i, and for some choice of the roots σ_i we have $\sum_{i=1}^{4} \sigma_i R_i \equiv 0$.*

A completely reducible cubic form F in 3 essential variables is equivalent under nonsingular linear transformations in the given field to $T = xyz$. By Theorem 2, $m(T) \leqq 4$. If $m(T)$ were 3, the form T would be equivalent to $C = au^3 + bv^3 + cw^3$ in the variables u, v, w, whence T is nonsingular. For T to be nonsingular it is necessary and sufficient[7] that the Hessian H of T split into linearly independent linear factors L, M, and N and under reduction of H to canonical form uvw, T transform covariantly to a reduced form C. Since the Hessian of T is already in canonical form and $T \not\equiv ax^3 + by^3 + cz^3$, we have $m(T) \neq 3$. The minimal number of a form cannot be less than the number of essential variables in the form, whence $m(T) = 4$. Hence $m(F) = 4$.

It is easy to prove that if $\sum_{i=1}^{r} \lambda_i (x + \alpha_i y)^n \equiv 0$, where the λ's are not zero, and $r \leqq n+1$, the α's can be grouped into sets $S_1, S_2, \cdots, S_\rho$ each of order 2 at least, where the α's in each set are equal; and if we let λ_i correspond to α_i, the sum of the λ's corresponding to the α's in S_i vanishes for each i in the range $1, 2, \cdots, \rho$. From this it follows rather immediately that if

$$6xyz \equiv \sum_{i=1}^{4} \lambda_i (x + \alpha_i y + \beta_i z)^3, \tag{2}$$

the right member of (2) is

$$\begin{aligned}(1/4ab)\{&(x + ay + bz)^3 - (x + ay - bz)^3 \\ &- (x - ay + bz)^3 + (x - ay - bz)^3\}.\end{aligned} \tag{3}$$

It is readily verified that the coefficients of x, y, and z in a representation $\lambda_i L_i^3$, $(i = 1, 2, 3, 4)$, of $6xyz$ are different from zero, whence any representation of $6xyz$ can be written as the right member of (2). Thus each representation of $6xyz$ is of the type (3), and (3) is a repre-

[7] Oldenburger, *Rational equivalence of a form to a sum of pth powers*, Transactions of this Society, vol. 44 (1938), pp. 219–249.

sentation of $6xyz$ for each choice of a, b not zero. Since the representations of each form equivalent to $6xyz$ under nonsingular transformations can be obtained from $6xyz$ by substitutions $x=L$, $y=M$, $z=N$ where L, M, N are linearly independent linear forms, *a cubic form F in 3 essential variables is completely reducible if and only if each 4-representation of F is of the type*

$$k\{(L + aM + bN)^3 - (L + aM - bN)^3 - (L - aM + bN)^3 + (L - aM - bN)^3\}, \tag{4}$$

where k, a, $b \neq 0$, and L, M, N are linearly independent.

Let a cubic form F in three essential variable be given by a minimal representation $\sum_{i=1}^{4} \mu_i R_i^3$. If F is completely reducible, the forms $\mu_i R_i^3$ (i not summed; $i=1, 2, 3, 4$) are identically equal to the forms $\pm k[L \pm aM \pm bN]^3$ in some order and for some choice of k, a, b, L, M, and N. Then there exists an element c in the given field K such that $\rho_i = (c\mu_i)^{1/3}$ are in K, and an ordering of the values of i so that

$$\begin{aligned} L + aM + bN &\equiv \rho_1 R_1, & L + aM - bN &\equiv -\rho_2 R_2, \\ L - aM + bN &\equiv -\rho_3 R_3, & L - aM - bN &\equiv \rho_4 R_4. \end{aligned} \tag{5}$$

Equations (5) are solvable for L, M, N if and only if $\sum_{i=1}^{4} \rho_i R_i \equiv 0$. Evidently there exists an element c in K so that roots ρ_i in K exist if and only if there exist roots $\sigma_i = (\mu_i/\mu_1)^{1/3}$ in K. Theorem 3 is now proved.

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