FACTORIZATION AND SIGNATURES OF LORENTZ MATRICES[1]

WALLACE GIVENS

The linear transformation $x \rightarrow \bar{x} = Lx$ (that is, $\bar{x}^i = L_j^i x^j$ with $i, j = 1, \cdots, n$) is called Lorentz if $(\bar{x}, \bar{x}) = (x, x)$, where

$$(1) \quad (x, x) = (x^1)^2 + (x^2)^2 + \cdots + (x^t)^2 - (x^{t+1})^2 - \cdots - (x^n)^2.$$

The matrices of all such transformations make up a group which we shall call the Lorentz group $\mathfrak{L}_{t,n-t}$. For $t=3$, $n=4$ it is usually called the extended Lorentz group.

In this paper we give extremely elementary proofs of two theorems. The first theorem has to do with the expression of a Lorentz matrix as a product of Lorentz matrices of simple type. For this it is sufficient that the elements of our matrices be chosen from a field of characteristic different from two. When the field is that of complex numbers, the signature of the quadratic form (1) is unimportant. The second theorem describes certain subgroups of the Lorentz group and for it we need an ordered (hence not a finite) field.

Each vector v for which $(v, v) \neq 0$ determines a transformation T_v with the equations

$$(2) \qquad \bar{x} = x - 2\frac{(v, x)}{(v, v)} v,$$

where

$$(3) \qquad (v, x) = v^1 x^1 + \cdots + v^t x^t - v^{t+1} x^{t+1} - \cdots - v^n x^n.$$

It is easy to verify the following:

(i) T_v is Lorentz.

(ii) The result of performing T_v twice is the identity.

(iii) Every multiple of v is a solution of the equations $\bar{x} = -x$, and conversely every solution is a multiple of v.

(iv) If $(v, y) = 0$ then $\bar{y} = y$ and conversely.

(v) For $v = e_j \equiv (0, \cdots, 0, 1, 0, \cdots, 0)$, where the one is in the jth place, T_{e_j} has the equations

$$(4) \quad \bar{x}^1 = x^1, \cdots, \bar{x}^{j-1} = x^{j-1}, \bar{x}^j = -x^j, \bar{x}^{j+1} = x^{j+1}, \cdots, \bar{x}^n = x^n.$$

(vi) If $v^1 = 0$, T_v has the equations

[1] Presented to the Society, November 25, 1938, under the title *Signatures of Lorentz matrices.*

$$\bar{x}^1 = x^1, \qquad \bar{x}^\alpha = M_\beta{}^\alpha x^\beta, \qquad \alpha, \beta = 2, \cdots, n, \tag{5}$$

where $\|M_\beta{}^\alpha\|$ is a matrix of $\mathfrak{L}_{t-1,n-t}$.

We call a transformation given by (2) a symmetry. Properties (iii) and (iv) show that each symmetry is a reflection in its associated hyperplane $(v, x)=0$ in the same way that (4) is a reflection in the coordinate hyperplane $x^i=0$.

THEOREM 1. *Every Lorentz transformation can be expressed as a product of symmetries.*[2]

PROOF. We use induction on n, the number of variables. Evidently the theorem is true for $n=1$. Moreover, by property (vi), the assumption that the theorem is true for $n-1$ variables enables us to conclude that it is also true for transformations of the form (5) on n variables.

We distinguish two cases according as $L_1^1 \neq 1$ or $L_1^1 = 1$ for the given Lorentz matrix L of order n.

In the first case the vector $v=e_1-Le_1$ determines a symmetry T_v (with matrix I_v), for $(v, v) = (e_1, e_1) - 2(e_1, Le_1) + (Le_1, Le_1) = \pm 2(1-L_1^1) \neq 0$, the minus sign occurring if $t=0$ in (1). Moreover, a computation with (2) shows that $I_vLe_1=e_1$; that is, $I_vL \equiv L^*$ is of such a form that $L_1^{*i}=0$ for $i>1$ and $L_1^{*1}=1$. Since L^* is Lorentz, $(e_1, e_j)=(L^*e_1, L^*e_j) = (e_1, L^*e_j) = \pm L_j^{*1}$, the minus sign occurring if $t=0$, and hence $L_j^{*1}=0$ for $j>1$. The transformation associated with L^* is therefore of the form (5). By the hypothesis of the induction we have $I_vL=L^*=I_1I_2 \cdots I_r$, where $I_1, \cdots,$ and I_r are the matrices of suitably chosen symmetries. Since $(I_v)^2=1$, $L=I_vI_1 \cdots I_r$ and we have the factorization of L.

In the second case we set $w=e_1+Le_1$ and observe that $(w, w) = \pm 2(1+L_1^1) \neq 0$. We find that $I_wLe_1 = -e_1$ and hence $L^* \equiv I_{e_1}I_wL$ (cf. (4)) is of the form (5). As before, $L^*=I_1 \cdots I_r$ and $L=I_wI_{e_1}I_1 \cdots I_r$. This completes the proof of Theorem 1.

We now assume that the field over which we are working is ordered and suppose that $0<t<n$, where t is the number of plus signs in (1).

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[2] P. F. Smith, *On the linear transformations of a quadratic form into itself*, Transactions of this Society, vol. 6 (1905), pp. 1–16, working with the complex field, proved this theorem and showed that n symmetries are sufficient. We referred to this paper in presenting this theorem to the Society on November 25, 1938. Since then we have seen a proof by E. Cartan (*Leçons sur la Théorie des Spineurs* I, pp. 13–17) that n symmetries suffice in the real case as well. Our proof is shorter and applies for all fields not of characteristic two, but gives a factorization into not more than $2n$ symmetries. L. Autonne in a long memoir, *Sur la décomposition d'une substitution linéaire* . . . , Annales de la Université de Lyons, 1903, solved the problem in the real definite case.

It will be evident that the following considerations do not give significant results in the definite cases $t=0$ and $t=n$.

Breaking up a Lorentz matrix L into partial matrices

$$L = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix}, \tag{6}$$

where A has t rows and columns, we find that the determinants $|A|$ and $|D|$ are different from zero. For, if $|A|=0$, there exists a vector $v=(v^1, \cdots, v^t, 0, \cdots, 0) \neq 0$ such that $Lv=\bar{v}=(0, \cdots, 0, \bar{v}^{t+1}, \cdots, \bar{v}^n)$. This is impossible for $(v, v)=(\bar{v}, \bar{v})$ and $(v, v)>0$ while $(\bar{v}, \bar{v})<0$. Similarly, $|D| \neq 0$.

We may therefore define for every Lorentz matrix the quantity

$$\sigma_+(L) = \begin{cases} +1 & \text{if } |A| > 0, \\ -1 & \text{if } |A| < 0. \end{cases} \tag{7}$$

Similarly, $\sigma_-(L)$ is defined in terms of $|D|$. Brauer and Weyl[3] have called these quantities the temporal and spatial signatures of L.

THEOREM 2. *For every pair of Lorentz matrices L and M,*

$$\sigma_+(L)\sigma_+(M) = \sigma_+(LM). \tag{8}$$

It will be sufficient to prove (8) only in the form

$$\sigma_+(L)\sigma_+(I) = \sigma_+(LI), \tag{9}$$

where I is the matrix of a symmetry. For, M can be written as a product of matrices of symmetries, and successive applications of (9) will give (8). Similar statements of course apply to $\sigma_-(L)$.

We shall need the following lemma:

LEMMA. *If $|A| \neq 0$ and rank $N=1$, then $|A-N|=|A|[1-\text{trace}\,(A^{-1}N)]$.*

PROOF. Since rank of $(A^{-1}N)$ is 1, $|\lambda A-N|=|A|\,|\lambda 1-A^{-1}N| =|A|[\lambda^n-\lambda^{n-1} \text{ trace } (A^{-1}N)]$, and we may take $\lambda=1$.

Let us call A, in (6), the spatial minor of L. Then the spatial minor of the matrix I_v of (2) is

$$1_t - \frac{2}{(v, v)} \|v^\alpha v^\beta\|, \tag{10}$$

[3] *Spinors in n dimensions*, American Journal of Mathematics, vol. 57 (1935), pp. 425–449. This paper includes a proof of our Theorem 2 (for the real field), using, however, the spinor representation of the Lorentz group. The theorem is so much simpler than the representation theory that it seems desirable to give a direct algebraic proof.

where 1_t is the unit matrix of order t and α and β have the range $1, \cdots, t$. If we write $(v^1)^2 + \cdots + (v^t)^2 = (v, v)_+$ and $(v^{t+1})^2 + \cdots + (v^n)^2 = (v, v)_-$ so that $(v, v) = (v, v)_+ - (v, v)_-$, the determinant of (10) is, by the lemma,

$$1 - \frac{2(v, v)_+}{(v, v)} = - \frac{(v, v)_+ + (v, v)_-}{(v, v)}. \tag{11}$$

Since $(v, v)_+ + (v, v)_- > 0$,

$$\sigma_+(I_v) = \begin{cases} -1 & \text{if } (v, v) > 0, \\ +1 & \text{if } (v, v) < 0. \end{cases} \tag{12}$$

Breaking up L as in (6) and applying the lemma to the determinant of the spatial minor of LI_v, we find it to be

$$- \frac{|A|}{(v, v)} Q(v), \tag{13}$$

where Q is a quadratic form in the variables $v^1, \cdots, v^n$ with the matrix

$$S = \begin{Vmatrix} 1_t & A^{-1}B \\ (A^{-1}B)' & 1_{n-t} \end{Vmatrix}, \tag{14}$$

where the prime denotes the tranpose matrix and the exponent -1 the inverse. To complete the proof it will be sufficient to show that $Q(v)$ is positive definite, for then the sign of $\sigma_+(LI_v)$ will be that of $-|A|/(v, v)$, and this, in virtue of (7) and (12), has the same sign as $\sigma_+(L)\sigma_+(I_v)$.

We prove that S is positive definite by showing that the matrix

$$T = \begin{Vmatrix} 1_t & -A^{-1}BD' \\ 0 & D' \end{Vmatrix} \tag{15}$$

transforms S into the identity; that is, $T'ST = 1$. To verify this we have recourse to the relations

$$A'A - C'C = 1_t, \qquad B'A - D'C = 0, \tag{16}$$

and

$$B'B - D'D = -1_{n-t},$$

which are necessary and sufficient in order that (6) be Lorentz. Computing $T'ST$ we find that it is 1_n if $K \equiv DD' - DB'(A')^{-1}A^{-1}BD'$ is equal to 1_{n-t}. This is proved by the computation

$$\begin{aligned} K &= DD' - DB'[1_t - (A')^{-1}C'CA^{-1}]BD' \\ &= DD' - DB'[1_t - BD^{-1}(D')^{-1}B']BD' \\ &= DD' - DB'BD' + DB'BD^{-1}(D')^{-1}B'BD' \\ &= DD' - D(D'D - 1_{n-t})D' + D(D'D - 1_{n-t})D^{-1}(D')^{-1}(D'D - 1_{n-t})D' \\ &= 2DD' - (DD')^2 + (DD' - 1_{n-t})(DD' - 1_{n-t}) \\ &= 1_{n-t}. \end{aligned}$$

Since $\sigma_+(1_n) = \sigma_-(1_n) = 1$, Theorem 2 gives $\sigma_+(L)\sigma_+(L^{-1}) = +1$ or $\sigma_+(L) = \sigma_+(L^{-1})$. Similarly, $\sigma_-(L) = \sigma_-(L^{-1})$. It follows that the set $\mathfrak{L}^{++}_{t,n-t}$ of Lorentz matrices satisfying the conditions

$$\mathfrak{L}^{++}_{t,n-t}: \qquad \sigma_+(L) = +1, \quad \sigma_-(L) = +1, \tag{17}$$

constitute a subgroup of $\mathfrak{L}_{t,n-t}$. Similarly, the Lorentz group contains the subgroups

$$\mathfrak{L}^{\pm+}_{t,n-t}: \qquad \sigma_+(L) = \pm 1, \quad \sigma_-(L) = +1, \tag{18}$$

$$\mathfrak{L}^{+\pm}_{t,n-t}: \qquad \sigma_+(L) = +1, \quad \sigma_-(L) = \pm 1. \tag{19}$$

The Lorentz matrices of determinant plus one of course also form a subgroup, say $\mathfrak{L}^{+}_{t,n-t}$. The matrices of this subgroup have the property that either

$$\mathfrak{L}^{+}_{t,n-t}: \quad \sigma_+(L) = \sigma_-(L) = +1 \text{ or } \sigma_+(L) = \sigma_-(L) = -1, \tag{20}$$

and every Lorentz matrix satisfying either of these conditions has determinant plus one. To prove this, we use (16) to get

$$\left\| \begin{matrix} A' & -C' \\ 0 & 1_{n-t} \end{matrix} \right\| \cdot \left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| = \left\| \begin{matrix} 1_t & 0 \\ C & D \end{matrix} \right\|. \tag{21}$$

Taking determinants gives $|A|\,|L| = |D|$. Since $|L| = \pm 1$, we have in all cases

$$|L| = \sigma_+(L)\sigma_-(L), \tag{22}$$

which justifies (20).

CORNELL UNIVERSITY