IMBEDDING THEOREMS IN DIFFERENTIAL GEOMETRY*

T. Y. THOMAS

The matters which I shall discuss today will be largely concerned with the general problem of imbedding a coordinate manifold of class C^r in a euclidean space of sufficiently many dimensions. I imagine you all have a fair idea as to what is meant by an n-dimensional manifold of class C^r. Briefly this may be described as a Hausdorff space each point of which admits a neighborhood homeomorphic to the interior of a sphere in an n-dimensional euclidean space. We then suppose that the coordinate systems which may be introduced into these neighborhoods by these homeomorphisms are such that the coordinate relations which exist in the intersection of two such coordinate neighborhoods are of class C^r, that is, possess continuous partial derivatives to the order r inclusive. Of course when we consider imbedding theorems in differential geometry, which is the title of this address, one usually thinks, possibly from historical reasons incidental to the development of the subject, that the given space is endowed with a Riemann metric, and is then concerned with the problem of isometric imbedding. We shall also have something to say about the problem of isometric imbedding, although in doing so we shall limit ourselves to results of a general character. Of necessity most of the mathematical details must be omitted from our discussion however interesting these may be, but here and there certain detailed considerations will be introduced when it appears that these are directly understandable and may be treated with dispatch.

Before proceeding to the discussion of our particular subject I should like to say a few words about the analogous purely topological imbedding problem. I have in mind principally the classical result of Menger [1] and Nöbeling [2] to the effect that every n-dimensional compact metric space is homeomorphic to a subset of the euclidean space of $2n+1$ dimensions. Let A and B be two compact metric spaces and let M denote the set of all continuous maps of A into subsets of B. If f and f' are two elements or points of M, we define the distance between these points to be the maximum value of the distance of the points $f(x)$ and $f'(x)$ as x runs over the points of the space A. With this definition of distance M becomes a metric space and may readily be shown to be complete, that is, every Cauchy sequence in M con-

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verges. Now Hurewicz [**3**] has given a very simple and elegant proof of the Menger-Nöbeling imbedding theorem based on his idea of the ϵ-map. An element f of the above metric space M is called an ϵ-map of A into B if every two points of A whose map points coincide in B have a distance less than ϵ from one another. Evidently if f is an ϵ-map for every $\epsilon>0$, it is a homeomorphic map of A into B and the intersection of any sequence of ϵ-maps with ϵ approaching zero is identical with the set of all homeomorphic maps of A into B. Now it may be shown that for any $\epsilon>0$ the ϵ-maps form an open set in M. Hurewicz then proceeds to show that if B is taken to be a complete sphere in the euclidean space of $2n+1$ dimensions, the set of all ϵ-maps for any value of $\epsilon>0$ is dense in the map space M. It follows by the Baire theorem that the compact metric space A can be mapped topologically into a subset of the $2n+1$ dimensional euclidean space and in fact that any continuous map of A into this euclidean space can be changed into a topological map by an arbitrarily small alteration.

I should now like to turn to the consideration of the work of Hassler Whitney [**4**] on the imbedding of coordinate manifolds in euclidean space. Owing to the extreme elegance of the treatment of the purely topological imbedding problem the methods there employed may very well serve as a model or perhaps as a goal toward which we may strive in our discussion of the imbedding of coordinate manifolds. I propose therefore to present some of the results of Whitney from this standpoint. First let me say that if the question is raised as to why the topological imbedding theorem does not suffice in the present case, the answer is the following: In the imbedding problem for coordinate manifolds of class C^r we demand more. We require in fact that the functions defining the imbedding shall be of class C^r and shall thus leave unaffected the underlying coordinate relationships which we recognize as a component part of the structure of the manifold.

In accordance with the above proposal let us impose the condition that our coordinate manifold is compact and metrizable. Now we know from point set theory that a compact Hausdorff space can be given a metric in the topological sense if, and only if, it is separable. We shall now show that a positive definite quadratic differential form can be defined over a compact and separable manifold of class C^r, the coefficients of this form being of class C^{r-1} as functions of the allowable coordinates of the manifold. Thus the coordinate manifold becomes a Riemann space of class C^{r-1} in the usual terminology.

It is easy to show the existence of a function $h(x)$ of class C^∞ defined for $-\infty < x < \infty$ such that

$$\text{(a)} \qquad h(x) = 0, \qquad x \leqq -1,\ x \geqq 1,$$

$$\text{(b)} \qquad h(x) = 1, \qquad -1/2 \leqq x \leqq 1/2,$$

$$\text{(c)} \qquad 0 < h(x) < 1, \qquad -1 < x < -1/2,\ 1/2 < x < 1.$$

Now let P be any point of our coordinate manifold and denote by $N(P)$ a coordinate neighborhood of P. We suppose the coordinates x^α so chosen in $N(P)$ that $x^\alpha = 0$ at P. By a coordinate transformation in $N(P)$ of the form $\bar{x}^\alpha = ax^\alpha$ (a = const.) we can so enlarge the coordinate representation of this neighborhood that it will contain a cube or box $-1 \leqq x^\alpha \leqq 1$. Denote the interior $(-1 < x^\alpha < 1)$ of this box by U and the box itself or the enclosure of U by $\bar{U}$. We shall also consider the smaller box $\bar{U}'$ defined by $-1/2 \leqq x^\alpha \leqq 1/2$ and its interior U'. In consequence of the above assumptions (compactness and separability) the coordinate manifold M is bicompact, that is, every covering of the manifold by open sets contains a finite covering. Hence a covering of the manifold M by the above open sets U' will contain a finite covering which we shall denote by $U_1', \cdots, U_t'$. Let us now put

$$H = h(x^1) \cdots h(x^n) \text{ in } \bar{U}, \qquad H = 0 \text{ in } M - \bar{U},$$

$$H_\alpha = x^\alpha H \text{ in } \bar{U}, \qquad H_\alpha = 0 \text{ in } M - \bar{U},$$

where U refers to any one of the coverings of the above finite set. Then $H = 1$ and $H_\alpha = x^\alpha$ in $\bar{U}'$. It is evident that the functions H and H_α are scalars of class C^r over M. Let us denote the functions H_α when determined in connection with the neighborhood U_1' by $\phi^1, \cdots, \phi^n$ and when determined in connection with the neighborhood U_2' by $\phi^{n+1}, \cdots, \phi^{2n}$, and so on. Then the equations

$$g_{\alpha\beta} = \sum_\nu \frac{\partial \phi^\nu}{\partial x^\alpha} \frac{\partial \phi^\nu}{\partial x^\beta}$$

define the components of a covariant tensor g of class C^{r-1} over M. Now the matrix $\|\partial\phi^\nu/\partial x^\alpha\|$ has rank n at every point P of M. In fact any point P is interior to one of the neighborhoods U_i' and in this neighborhood $H_\alpha = x^\alpha$. Hence the matrix $\|\partial\phi^\nu/\partial x^\alpha\|$ must contain the diagonal determinant $|\delta_\beta{}^\alpha|$ at P. Thus at any point P of M the above functions $g_{\alpha\beta}$ appear as the coefficients of a positive definite quadratic differential form by which a Riemann metric is defined in M.

Conversely if a (connected) compact manifold M admits as above a Riemann metric g, we may metrize M in the topological sense by defining the distance $\rho(P, Q)$ of two points P and Q as the greatest lower bound of the lengths of all curves of class C^r (broken or continuously differentiable) which join P to Q. It follows that M is separable. The result obtained may perhaps be stated as the following theorem.

RIEMANN METRIZATION THEOREM. *Any compact manifold M of class C^r with $r \geqq 1$ admits a Riemann metric of class C^{r-1} if, and only if, M is separable.*

From now on we shall consider a fixed metric g of class C^{r-1} in M which exists in accordance with the above theorem. When it is desired to measure distances independently of coordinate systems this metric will be used. Within the separate coordinate systems the euclidean metric may however sometimes be employed to advantage.

Let f be a map of class C^r of M into the euclidean space E_m. The dimensionality m of the euclidean space will not be fixed for the present but will be determined later on the basis of our discussion. Let S be the set of all such maps f. We define a metric in S in the following manner: If ϕ and ψ are two elements of S, let us put

$$
\begin{aligned}
&d_0(\phi(x), \psi(x)) = \left\{ \sum [\phi^i(x) - \psi^i(x)][\phi^i(x) - \psi^i(x)] \right\}^{1/2},\\
&d_1(\phi(x), \psi(x)) = \left\{ \sum g^{\alpha\beta}[\phi^i_{,\alpha} - \psi^i_{,\alpha}][\phi^i_{,\beta} - \psi^i_{,\beta}] \right\}^{1/2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,\\
&d_r(\phi(x), \psi(x)) = \left\{ \sum g^{\alpha\mu} \cdots g^{\beta\nu}[\phi^i_{,\alpha\cdots\beta} - \psi^i_{,\alpha\cdots\beta}][\phi^i_{,\mu\cdots\nu} - \psi^i_{,\mu\cdots\nu}] \right\}^{1/2},
\end{aligned}
$$

where it is to be understood that Latin indices have the range $1, \cdots, m$ and that Greek indices have the range $1, \cdots, n$ and both sets of indices are to be summed when repeated in accordance with the usual convention. Note also that $d_1, \cdots, d_r$ involve successive covariant derivatives of the functions ϕ and ψ and that d_r involves the rth covariant derivative which is the highest covariant derivative that can be formed under the hypothesis that the metric g is of class C^{r-1} and the maps ϕ and ψ are of class C^r. Put

$$D(\phi(x), \psi(x)) = d_0(\phi(x), \psi(x)) + \cdots + d_r(\phi(x), \psi(x)).$$

Then $D(\phi(x), \psi(x))$ is a continuous function on M. Since M is bicompact, the function $D(\phi(x), \psi(x))$ assumes its maximum value at a point of M. Denote this maximum value by $D(\phi, \psi)$ and define $D(\phi, \psi)$ as the distance between the points ϕ and ψ of S. It is easily seen that with this definition of distance S is a metric space. Moreover it can readily be shown that S is complete. The fact that the map space S whose elements are the maps of class C^r of the coordinate manifold M into the euclidean space E_m thus appears as a complete metric space is in strict analogy with the situation in the topological imbedding problem, and imbedding theorems which are likewise analogous to those of the topological theory can be demonstrated.

We shall say that a map or point ϕ in S is *regular* if the matrix $\|\partial\phi^i/\partial x^\alpha\|$ has rank n at every point of M. Otherwise ϕ will be said to be *singular* and the points at which the above matrix has rank less than n will be called the *singular points* of the map. The set of all such singular points will be called the domain of *singularity* of the map ϕ.

As one can readily imagine, there are many details in this demonstration which have no counterpart in the purely topological theory. It seems inadvisable to enter into such details here especially in view of the fact that the main results can be stated directly in terms of the point of view which we have now established and when so stated are immediately understandable. We shall therefore content ourselves here with the following statement of what may be classified as two of the main imbedding theorems for coordinate manifolds:

I. *If $r \geqq 2$ and $m \geqq 2n$, the regular maps form an open and dense set in S.*

II. *If $r \geqq 2$ and $m \geqq 2n+1$, the regular topological maps form an open and dense set in S.*

Since $f=0$ is an element of S, it follows that the sets which enter into the above theorems are nonvacuous. Thus any compact and separable manifold of class C^r with $r \geqq 2$ can be imbedded by a regular topological map of class C^r in the euclidean space of $2n+1$ dimensions and an infinitesimal alteration in any map of class C^r will result in a regular topological map in strict analogy with the topological imbedding theorem (Whitney [**4**]).

There is perhaps a mild interest in considering the extreme case in which the imbedding of the coordinate manifold is in the euclidean space E_n and thus of the same dimensionality as the manifold itself. Assuming as above that the manifold is compact and separable and of $n \geqq 2$ dimensions, it suffices now to suppose that it is of class C^r with $r \geqq 1$. We now let S be the complete metric space whose elements are the maps of class C^r of the manifold M into the euclidean space E_n and denote by Σ the set of all points in S which correspond to maps whose domain of singularity is nowhere dense in M. It may then be proved that the following theorem holds.

III. *The set Σ is dense in S.*

It follows that the manifold M can be imbedded in the euclidean space of n dimensions by a map of class C^r whose domain of singularity is closed and nowhere dense in M. In particular it is thus possible to define a Riemann metric of class C^{r-1} in M which is nonsingular and locally flat in the ordinary sense except over a closed and nowhere dense set in the space (Thomas [5]).

The methods and results which we have so far considered cannot be applied to the case of the analytic manifold. Nor shall we now have much to say about such manifolds since as far as I am aware there are no results of a general character pertaining to the imbedding of analytic manifolds in euclidean space. Recently Bochner [6] has shown that a compact and separable analytic manifold which bears an analytic Riemann metric, that is, a compact analytic Riemann space, can be imbedded analytically and topologically in a euclidean space of $2n+1$ dimensions. Bochner's demonstration consists in showing that it is possible to approximate a map of class C^r with $r \geqq 2$ of the manifold in the euclidean space as closely as desired by an analytic map. Selecting the map of class C^r to be topological, it follows that not only will the analytic map exist but it will likewise be topological. Yet it must be confessed that this result does not constitute a strict imbedding theorem for analytic manifolds owing to the assumption of the existence of the Riemann metric. In fact under this assumption one would naturally inquire into the question of the isometric imbedding (in the differential sense) of the Riemann space into euclidean space. Some time ago I considered the question of the possibility of joining two or more arbitrary points of an analytic manifold by an analytic arc and was able to accomplish this only under the assumption that the manifold possessed an analytic affine connection [7]. While the assumption of the existence of the affine connection is presumably somewhat weaker than Bochner's assumption of the Riemann metric, it is nevertheless an assumption of analogous character. Apparently one is here faced with an underlying difficulty the essential idea for the solution of which remains as yet undiscovered.

Let us now turn our attention to the problem of the isometric imbedding of a Riemann space in euclidean space to which in fact our discussion has naturally led. This problem may be identified with the problem of finding a solution $\phi_1, \cdots, \phi_m$ of the system

$$g_{\alpha\beta} = \sum_{i=1}^{m} \frac{\partial \phi_i}{\partial x^\alpha} \frac{\partial \phi_i}{\partial x^\beta}$$

over the Riemann space with fundamental metric tensor g. The result that an n dimensional Riemann space can be imbedded locally and isometrically in a euclidean space of $n(n+1)/2$ dimensions seems to have been enunciated first by Schlaefli [8]. Janet [9] is usually considered the first to have made a serious attempt to prove the local isomorphic imbedding theorem for Riemann spaces, the result at which he arrived being that above stated. This was followed by proofs of the same theorem by Cartan [10] and Burstin [11]. In all cases the theorem in question was made to depend on more or less standard results in the theory of systems of differential equations and so need not be considered further on this occasion.

With regard to the question of the isomorphic imbedding of Riemann spaces in the large in euclidean space no general result seems to be known. It would appear therefore that here one would find an interesting although difficult field for investigation.

It is of course well known that the condition for the local isomorphic imbedding of a Riemann space of n dimensions in the euclidean space of n dimensions is that the curvature tensor shall vanish over the Riemann space. If furthermore the given Riemann space is simply connected, this condition suffices for its complete imbedding in the n dimensional euclidean space.

The problem of determining the conditions under which an n dimensional Riemann space is of class one, that is, can be imbedded isometrically in a euclidean space of $n+1$ dimensions but not in an n dimensional euclidean space, admits in general a solution of some degree of refinement. As I have occasion to believe that this imbedding theorem is not generally well known I should like to indicate in slight detail at least some of the essential features on which the solution of this problem depends.

Let R denote a Riemann space of class C^2. It is then well known that the following system of equations,

$$\frac{\partial y^i}{\partial x^\alpha} = y^i_{,\alpha}, \quad \frac{\partial y^i_{,\alpha}}{\partial x^\beta} = \sum_{\nu=1}^{n} \Gamma^\nu_{\alpha\beta} y^i_{,\nu} + b_{\alpha\beta}\sigma^i, \quad \frac{\partial \sigma^i}{\partial x^\alpha} = \sum_{\mu,\nu=1}^{n} b_{\alpha\mu} g^{\mu\nu} y^i_{,\nu},$$

constitutes necessary conditions for the isomorphic imbedding of R in $n+1$ dimensional euclidean space, that is, for R to appear as a hypersurface in the euclidean space. In these equations the y^i denote the coordinates of the euclidean space, the Γ's are the components of the Christoffel symbols of the Riemann space R, the b's are the coefficients of the second fundamental form of the hypersurface and the σ's are the components of the vectors in the euclidean space normal to the hypersurface. As integrability conditions of the above system we have

$$b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta} \qquad \text{(Codazzi equations)},$$

$$B_{\alpha\beta\gamma\delta} = b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta} \qquad \text{(Gauss equations)},$$

in which the first set of these equations involves the components of the covariant derivative of the above tensor b and the second set of equations contains in its left member the components of the completely covariant form of the curvature tensor of the Riemann space R. In the above general form these equations were first obtained by Voss. They are usually however referred to as the Gauss and Codazzi equations since they are equivalent to conditions originally found by Gauss and Codazzi for the special case of two dimensional surfaces. The usual isomorphic imbedding theorem for Riemann spaces of class one can now be stated as follows: *An open and simply connected coordinate neighborhood U of the above Riemann space R can be imbedded isomorphically in the $n+1$ dimensional euclidean space if, and only if, the Gauss and Codazzi equations are satisfied in U.* It is well known too that the set of quantities $b_{\alpha\beta}$ satisfying the Gauss and Codazzi equations in the neighborhood U appear as the coefficients of the second fundamental form of the hypersurface which exists by the above theorem and that when these quantities $b_{\alpha\beta}$ are fixed the hypersurface is determined to within a motion in the euclidean space.

The above theorem cannot be considered to give a solution of the local isometric imbedding problem in any fundamental sense. For the conditions in question are of differential character and in this respect are analogous to the conditions by which the imbedding itself is defined. On the other hand the condition for a Riemann space to be locally flat or to be capable of being imbedded locally in a euclidean space of the same dimensionality is expressible by the vanishing of a pure invariant, namely the curvature tensor of the given Riemann space. It is likewise possible to express conditions for a Riemann space to be of class one in terms of the behavior of its intrinsic invariants and I should now like to indicate some of the steps by which these conditions can be established.

Let us say that a hypersurface of the euclidean space of $n+1$ dimensions is of type one if the rank of the matrix of the coefficients of the second fundamental form is zero or one and that the hypersurface is of type τ where τ is an integer of the set $2, \cdots, n$, if the rank of this matrix is τ over the hypersurface. As so defined the type number is not an intrinsic invariant but depends upon the relation of the space to the euclidean space in which it is imbedded. It can be shown however that the type number of a hypersurface is determined by its intrinsic properties, that is, by the first fundamental form. As a consequence we can speak of the type number of a Riemann space R of class C^2 regardless of whether or not this space can be considered as a hypersurface of the euclidean space. Let us also say that a hypersurface is *intrinsically rigid* if the second fundamental form is uniquely determined (to within algebraic sign) by the first fundamental form and the equations of Gauss and Codazzi. It follows that an intrinsically rigid hypersurface cannot be subjected to a continuous deformation in the euclidean space without altering its internal metric properties. The following theorem can also be proved: *A hypersurface of type* $\tau \geqq 3$ *is intrinsically rigid.*

The solution of the problem before us depends essentially on the happy circumstance that under certain rather general conditions the equations of Codazzi are consequences of the equations of Gauss. In fact it can be shown that if R is any Riemann space of class C^2 and type not less than 4 and if there exists a set of symmetric quantities $b_{\alpha\beta}$ of class C^1 in a coordinate neighborhood U which satisfy the equations of Gauss, then the equations of Codazzi will automatically be satisfied in U. With this result the determination of the conditions for a Riemann space to be of class one is reduced essentially to an algebraic problem. It can be shown that the Gauss equations considered as equations for the determination of the symmetric quantities $b_{\alpha\beta}$ admit a resultant system and to this system further algebraic conditions can be added in the form of inequalities which are both necessary and sufficient for the reality of the solutions. Assuming that the Riemann space under consideration is of class C^2 and that the components of its curvature tensor are continuous functions, which is somewhat weaker than the requirement that the space be of class C^3, the general result at which one arrives is the following: *There exist sets of polynomials* F_1, F_2 *and* F_3 *in the components of the curvature tensor such that a simply connected n-dimensional Riemann space of type not less than* 4 *can be imbedded isometrically in the euclidean space of* $n+1$ *dimensions if, and only if,* $F_1>0$, $F_2\geqq 0$ *and* $F_3=0$ *over the space* (Thomas [**12**]). For want of a better name I have said that such conditions constitute an *algebraic characterization.* With some modifications in procedure it is possible to extend the above result to Riemann spaces of type 3 and in fact to spaces of variable type. Moreover the type of a Riemann space is itself capable of being characterized algebraically and so may be included if desired in the algebraic characterization. It must be emphasized however that without the above restriction on the type of the Riemann space these algebraic procedures are not applicable and indeed it is questionable if the class of Riemann spaces which can be imbedded isometrically in the euclidean space of one higher dimension admits an algebraic characterization.

These results have recently been extended by C. B. Allendoerfer [**13**] to Riemann spaces of class greater than one although the methods exhibit considerable formal complication owing without doubt to the nature of the problem.

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University of California at Los Angeles