PROPERTIES OF GENERALIZED DEFINITIONS OF LIMIT*

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1. **Introduction.** The theory of summability has been the subject of several excellent expository addresses† presented to this Society. These addresses have dealt largely with properties of matrix transformations

$$A: \qquad y_s = \sum_{t=0}^{\infty} A_{s,t} x_t,$$

which associate with certain sequences $x_0, x_1, x_2, \cdots$ of complex numbers the sequences $y_0, y_1, \cdots$ determined by use of a given matrix $A_{s,t}$ of complex constants.

It is my object to discuss, and to compare with the matrix transformations A, the kernel transformations

$$K: \qquad y(s) = \int_0^{\infty} K(s, t) x(t) dt,$$

which associate with certain complex-valued functions $x(t)$ defined over $0 < t < \infty$ the functions $y(t)$ determined by a given kernel $K(s, t)$ belonging to a certain class of complex-valued functions which we specify in §3. Transformations of this form were first studied by Silverman.‡ More recent contributions§ have been made by Knopp, Hill, Raff, and Day.

The point of view of the present study of kernel transformations is quite different from that of earlier ones. The earlier studies have started with either the Riemann or Lebesgue integral and the class X

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† W. B. Ford, this Bulletin, vol. 25 (1918–1919), pp. 1–15; R. D. Carmichael, ibid., vol. 25 (1918–1919), pp. 97–131; C. N. Moore, ibid., vol. 25 (1918–1919), pp. 258–276; W. A. Hurwitz, ibid., vol. 28 (1922), pp. 17–36; and C. N. Moore, ibid., vol. 37 (1931), pp. 240–250.

‡ L. L. Silverman, *On the notion of summability for the limit of a function of a continuous variable*, Transactions of this Society, vol. 17 (1916), pp. 284–294.

§ K. Knopp, *Zur Theorie der Limitierungsverfahren*, Mathematische Zeitschrift, vol. 31 (1929–1930), pp. 97–127; pp. 276–305. J. D. Hill, *A theorem in the theory of summability*, this Bulletin, vol. 42 (1936), pp. 225–228. H. Raff, *Lineare Transformationen beschrankter integrierbarer Funktionen*, Mathematische Zeitschrift, vol. 41 (1936), pp. 605–629; *Über lineare Integraltransformationen*, Monatshefte für Mathematik und Physik, vol. 45 (1937), pp. 379–393. M. M. Day, *Regularity of function-to-function transformations*, this Bulletin, abstract 44-9-332.

of all functions $x(t)$ which are bounded and integrable over each finite interval; and the fundamental problem solved is that of characterizing the functions $K(s, t)$ defining transformations *regular over the assigned set* X, that is, transformations such that each $x \in X$ for which $\lim_{t\to\infty} x(t)$ exists has a transform $y(s)$ for which $\lim_{s\to\infty} y(s) = \lim_{t\to\infty} x(t)$. The papers of Knopp are apparent exceptions; but later writers have implied that Knopp should have introduced the set X to make his work precise. The present study starts with a definition of integral (see §2) and a kernel $K(s, t)$ belonging to a certain class of functions (see §3); and the fundamental problem which presents itself is that of setting up criteria to determine what properties the transformation thereby determined has or fails to have.

This address is entirely self-contained in the sense that no knowledge of either the now extensive known theory of matrix transformations, or the more modest known theory of kernel transformations, is assumed, and that all proofs are given in terms of fundamental notions of analysis. That this is so is not purely a recognition of the fact that this address should be so constructed to meet the needs of optimistic individuals who, without previous experience with the theory of summability, might hope to gain from this address some knowledge of the theory. The author has felt for years that those who work in the theory of summability (and in particular the author himself) should have in print a self-contained foundation for further work in the theory of kernel transformations.

Some examples and remarks indicate the manner in which we begin to develop de novo the theory of kernel transformations. Two of the simplest and most useful transformations of the forms A and K respectively are

$$y_s = \sum_{t=0}^{s} \frac{1}{s+1} x_t, \qquad s = 0, 1, 2, \cdots, \tag{1.01}$$

in which $A_{st} = 1/(s+1)$ or 0 according as $0 \leqq t \leqq s$ or $t > s$; and

$$y(s) = \int_0^s \frac{1}{s} x(t)dt, \qquad s > 0, \tag{1.02}$$

in which $K(s, t)$, defined for $s > 0$, is $1/s$ or 0 according as $0 \leqq t \leqq s$ or $t > s$. We shall emphasize later the point that the transformation (1.02) does not become meaningful until the definition of integral used there has been specified. Let us use for the moment either the Lebesgue integral or the "improper" Cauchy-Riemann integral. If we let the function $y(s)$ given by (1.02) be denoted by $y_1(s)$, and let the

(1.02) transform of $y_{r-1}(s)$ be denoted by $y_r(s)$, it can easily be shown by induction that, for each $r=1, 2, \cdots$,

$$(1.03)\qquad y_r(s) = \frac{1}{\Gamma(r)s}\int_0^s \left(\log\frac{s}{t}\right)^{r-1} x(t)dt.$$

As a matter of fact (1.03) defines, for each complex r having a positive real part, a transformation (the Hölder transformation of order r) with kernel

$$(1.04)\qquad \begin{aligned} K(s, t) &= [\log (s/t)]^{r-1}/\Gamma(r)s, && 0 < t < s, \\ &= 0, && t \geqq s. \end{aligned}$$

The orthodox (ϵ, δ) definition of limit is, from a sufficiently abstract point of view, one scheme for associating with each sequence x_s, or function $s(x)$ belonging to a certain class, a number L called its limit. The transformations A and K furnish *generalized definitions of limit* or *methods of summability* when one defines $\lim_{s\to\infty} y_s$ and $\lim_{s\to\infty} y(s)$ to be generalized limits of a sequence x_s and a function $x(s)$, respectively, when the limits exist. We conform to accepted terminology in calling x_s *summable* A to L in case $y_0, y_1, \cdots$ exist and $\lim_{s\to\infty} y_s = L$, and $x(s)$ *summable** K to L in case $y(s)$ exists for $s>0$ and $\lim_{s\to\infty} y(s) = L$. Thus each method of summability furnishes, as does the (ϵ, δ) definition of limit, a scheme of associating with certain sequences or functions numbers which may be called their "limits."

A transformation A (or matrix A) is called regular if each convergent sequence x_t is summable A to the value to which it converges. For example, it is well known and is a good exercise for undergraduates to show that (1.01) is regular. Formulation of a useful definition of regularity of K is not quite so simple, and is postponed to §5.

Let $A_{s,t}$ be a matrix and let x_t and y_s be sequences so related that

$$(1.05)\qquad y_s = \sum_{t=0}^{\infty} A_{s,t} x_t, \qquad s = 0, 1, 2, \cdots.$$

If step functions $x(t)$ and $y(s)$ and a step kernel $K(s, t)$ are defined by the equations

$$(1.06)\qquad x(t) = x_{[t]},\quad y(s) = y_{[s]},\quad K(s, t) = A_{[s],[t]},$$

where $[r]$ denotes the greatest integer less than or equal to r, then

* It is of course possible to modify these definitions, calling x_s or $x(s)$ summable A or K to L in case y_s or $y(s)$ exist for all sufficiently great s and $\lim_{s\to\infty} y_s = L$ or $\lim_{s\to\infty} y(s) = L$. These modified definitions, which turn out to be not significantly different from ours, are discussed at the end of §9.

under any one of several definitions of integral (1.05) can be written

$$(1.07) \qquad y(s) = \int_0^\infty K(s, t)x(t)dt, \qquad s > 0.$$

This means simply that any matrix transformation A can be represented as a kernel transformation K in which the domain and range are limited to step functions constant over each interval $n \leqq t < n+1$, $n = 0, 1, \cdots$. On account of this fact (which will be discussed in more detail in §10), an advance in the theory of matrix transformations suggests at least the possibility of making a corresponding advance in the theory of kernel transformations.

The theory of kernel transformations K has lagged far behind the theory of matrix transformations A. This is unquestionably due in part to the fact that the theory of K is really more difficult than that of A. The author feels that this is also due in part to the fact that in spite of much work on linear transformations in general and kernel transformations in particular, there never has been an adequate foundation laid for development of the theory of K analogous to recent developments in the theory of A.

When we compare a kernel transformation K with a matrix transformation A, a fundamental difference between the two appears immediately. On the one hand,

$$(1.08) \qquad \sum_{t=0}^{N} A_{s,t}x_t = A_{s,0}x_0 + A_{s,1}x_1 + \cdots + A_{s,N}x_N$$

always exists and has a unique meaning for all mathematicians when a matrix A, a sequence $x_1, x_2, \cdots$ of complex numbers, and an integer $N \geqq 0$ are given. On the other hand, specification of a kernel $K(s, t)$, a function $x(t)$, and a number $h > 0$ is not (in these days when a multiplicity of different definitions of integral are used in analysis) sufficient to determine whether

$$(1.09) \qquad \int_0^h K(s, t)x(t)dt$$

exists. Not only the fact of existence of (1.09) but also the value of (1.09) may depend upon the definition of integral used.

It is neither economical nor satisfying to develop the theory of kernel transformations first for Riemann integrals, then for Lebesgue integrals, then for one or more other integrals, and then perhaps finally for general integrals of Banach type under which each bounded function is integrable over each bounded set. Moreover an attempt to study kernel transformations without prescribing the type or the properties of the integrals involved is utterly futile and meaningless.

It turns out that a considerable part of a theory of K analogous to known theory of A can be developed for any definition of integral having the eight properties which we give in §2. In §3, we define *kernel* and in §4 we discuss the class $\mathfrak{K}$ of functions $x(t)$ such that the integral

$$\int_0^h K(s, t)x(t)dt$$

exists for each pair of positive numbers h and s. In §5, regularity is defined and discussed. In §§6–8, we give several theorems which supply the tedious parts of proof of necessity for theorems on regularity, and so on, in §9. The theorems of §§6–8 are made sufficiently general to furnish proof of necessity for many other theorems in the theory of summability which we shall be unable to give in a paper of temperate length. In §10, transformations whose kernels are step kernels are related to sequence-to-function and matrix transformations. In §11, the scope of regular transformations is discussed briefly. Finally in §12 we indicate the possibility of taking point sets other than the set of positive numbers for the domains of s and t in kernel transformations.

2. **Properties of integral.** In the future we use those and only those definitions of integral having properties which we now specify. To simplify our statements of the properties, we use the symbol "$f \varepsilon I(a, b)$" to indicate that a and b are finite real numbers with $a<b$ and that $f(t)$ is a member of the class $I(a, b)$ of complex-valued functions integrable over the interval $a \leqq t \leqq b$.

I. *If* $f_1, f_2 \varepsilon I(a, b)$ *and* c_1, c_2 *are complex constants, then* $c_1f_1+c_2f_2 \varepsilon I(a, b)$ *and*

$$(2.1) \qquad \int_a^b [c_1f_1(t) + c_2f_2(t)]dt = c_1\int_a^b f_1(t)dt + c_2\int_a^b f_2(t)dt.$$

II. *If* f_1 *and* f_2 *are real, then* $f_1+if_2 \varepsilon I(a, b)$ *if and only if* $f_1 \varepsilon I(a, b)$ *and* $f_2 \varepsilon I(a, b)$.

III. *If* $f \varepsilon I(a, c)$ *and* $a<b<c$, *then* $f \varepsilon I(a, b)$, $f \varepsilon I(b, c)$, *and*

$$(2.2) \qquad \int_a^c f(t)dt = \int_a^b f(t)dt + \int_b^c f(t)dt.$$

If $a<b<c$, $f \varepsilon I(a, b)$, *and* $f \varepsilon I(b, c)$, *then* $f \varepsilon I(a, c)$ *and* (2.2) *holds.*

IV. *If $a \leqq b$, then*

$$\int_a^b 1dt = b - a. \tag{2.3}$$

V. *If $f_1, f_2 \in I(a, b)$ while f_1 and f_2 are real and $f_1(t) \leqq f_2(t)$ over $a \leqq t \leqq b$, then*

$$\int_a^b f_1(t)dt \leqq \int_a^b f_2(t)dt. \tag{2.4}$$

This implies that if $f \in I(a, b)$ and f is real, then the integral of f is real.

VI. *If f, $|f| \in I(a, b)$, then*

$$\left| \int_a^b f(t)dt \right| \leqq \int_a^b | f(t) | \, dt. \tag{2.5}$$

VII. *If, for each h with $a < h < b$, f, $|f| \in I(a, h)$ and*

$$\int_a^h | f(t) | \, dt < M, \tag{2.6}$$

M being a constant independent of h, then $f \in I(a, b)$ and

$$\int_a^b f(t)dt = \lim_{h \to b-} \int_a^h f(t)dt; \tag{2.7}$$

likewise, if $f \in I(h, b)$ when $a < h < b$ and

$$\int_h^b | f(t) | \, dt < M, \tag{2.8}$$

M being a constant independent of h, then $f \in I(a, b)$ and

$$\int_a^b f(t)dt = \lim_{h \to a+} \int_h^b f(t)dt; \tag{2.9}$$

except that, for any integral (and in particular for the Riemann integral) for which $f \in I(a, b)$ implies that $f(t)$ must be bounded over $a \leqq t \leqq b$, VII *shall apply only to functions $f(t)$ bounded over $a \leqq t \leqq b$.*

VIII. *The left member of*

$$\int_a^\infty f(t)dt = \lim_{h \to \infty} \int_a^h f(t)dt \tag{2.91}$$

is defined to be the right member whenever the right member exists.

Any integral satisfying I–VIII will be denoted hereafter by $\int$.

All of the conditions I to VII (that is, those which pertain to finite intervals) are satisfied by both Riemann and Lebesgue integrals. These integrals, extended over $(0, \infty)$ by (2.91), will be called simply Riemann and Lebesgue integrals respectively; hence Riemann and Lebesgue integrals satisfy I–VIII. Existence of

$$\int_a^\infty f(t)dt \tag{2.92}$$

does not imply existence of

$$\int_a^\infty |f(t)|dt. \tag{2.93}$$

However, properties I, III, V, VI, VIII, and the Cauchy criterion for convergence imply that, if $f \varepsilon I(a, h)$ for each $h > a$ and (2.93) exists, then (2.92) must exist.

Properties III, V, VI, and VII imply that $\int$ shares with Riemann and Lebesgue integrals the property of being independent of the value of the integrand at any finite set of points. For example, if $f \varepsilon I(a, b)$ and $g(t) = f(t)$ for $a < t < b$, then $g \varepsilon I(a, b)$ and

$$\int_a^b g(t)dt = \int_a^b f(t)dt$$

irrespective of whether $g(a)$ and $g(b)$ agree with $f(a)$ and $f(b)$ or are different from $f(a)$ and $f(b)$. Again, if $f \varepsilon I(a, b)$ and $g \varepsilon I(b, c)$ where $a < b < c$, and $h(x) = f(x)$ or $g(x)$ according as $a \leqq x \leqq b$ or $b < x \leqq c$, then $h \varepsilon I(a, c)$ and

$$\int_a^c h(x)dx = \int_a^b f(x)dx + \int_b^c g(x)dx.$$

Properties of $\int$ imply that, if f is real and bounded and $f \varepsilon I(a, b)$, then the integral over (a, b) of f lies in the closed interval bounded by the lower and upper Darboux (sometimes called Riemann) integrals over (a, b) of f.

3. **Definition of kernel.** We next decide what kind of functions $K(s, t)$ we shall admit as kernels in the transformation

$$K: \qquad y(s) = \int_0^\infty K(s, t)x(t)dt.$$

Suppose the integral chosen happens to be a Riemann integral. It is then true that, if $x(t)\equiv 0$,

$$\int_0^h K(s, t)x(t)dt, \qquad h > 0, \tag{3.01}$$

will exist for every function $K(s, t)$; if, however, one wishes (3.01) to exist for all functions $x(t)$ in some significant class of functions, it would be at least unpleasant to have a function $K(s, t)$ which is not a Riemann integrable function (say unbounded or perhaps nonmeasurable) of t over $0\leqq t\leqq h$ for each s. In other words, it seems reasonable to impose some "condition of integrability" on $K(s, t)$ to ensure that the transformation K will be significant. Accordingly we adopt the following definition.*

A complex-valued function $K(s, t)$ will be called a *kernel* if the integrals

$$\int_0^h K(s, t)dt, \qquad h, s > 0, \tag{3.02}$$

$$\int_0^h K(s_1, t) \operatorname{sgn} K(s_2, t)dt, \qquad h, s_1, s_2 > 0, \tag{3.03}$$

$$\int_0^h K(s_1, t)\Re \operatorname{sgn} K(s_2, t)dt, \qquad h, s_1, s_2 > 0, \tag{3.04}$$

all exist. Existence of (3.03) and (3.04) implies existence of

$$\int_0^h K(s_1, t)\Im \operatorname{sgn} K(s_2, t)dt, \qquad h, s_1, s_2 > 0. \tag{3.05}$$

Setting $s_1=s_2=s$ in (3.03) gives existence of

$$\int_0^h | K(s, t) | dt, \qquad h, s > 0. \tag{3.06}$$

The class of functions $K(s, t)$ satisfying (3.02), (3.03), and (3.04) is the class referred to in §1. Accordingly K denotes in this paper either a function satisfying (3.02), (3.03), and (3.04), or the transformation determined by such a kernel.

For the case of Lebesgue integrals, existence of (3.03) and (3.04) need not be explicitly required since it is implied by existence of (3.02). For the case of Riemann integrals, however, existence of (3.02) does not imply that of (3.03) and (3.04). The latter fact is readily

* If u and v are real and $z=u+iv$, then $\Re z=u$, $\Im z=v$; and sgn z is 0 or $|z|/z$ according as $z=0$ or $z\neq 0$, so that in every case $z \operatorname{sgn} z=|z|$.

proved by use of the function $g(t)$ defined as follows. Let $r_1, r_2, \cdots$ denote in some order the positive rational numbers, and let $g(t)=r_n/n$ or 0 according as $t=r_n$ or t is irrational. In each interval $(0, h)$, $g(t)$ has the Riemann integral 0 while sgn $g(t)$ is not Riemann integrable. Thus we have here one more place in analysis where a theory based on use of Lebesgue integrals avoids assumptions [in this case (3.03) and (3.04)] which are required for a corresponding theory based on Riemann integrals.

The conditions (3.02), (3.03), and (3.04) do not (unless $\int$ happens to be of a restrictive type) imply that $K(s, t)$ is bounded over each interval $0<t<h$. For example, if $\mathcal{R}r\neq 1$ the function (1.04) associated with the Hölder transformation (1.03) of order r, being unbounded over $0<t<s$ for $s>0$, would not be a kernel when $\int$ is the proper Riemann integral; but when $\int$ is either the Cauchy-Riemann improper integral or the Lebesgue integral, the function is a kernel when $\mathcal{R}r>0$.

4. **The class $\mathcal{K}$.** If a function x is such that

$$y(s) = \int_0^\infty K(s, t)x(t)dt \tag{4.01}$$

exists for each $s>0$, then conditions VIII and III of §2 imply that

$$F(h, s) = \int_0^h K(s, t)x(t)dt \tag{4.02}$$

must exist for each $h, s>0$. Let $\mathcal{K}$ be the class of functions x for which $F(h, s)$ exists for $h, s>0$. The condition $x \in \mathcal{K}$ is essentially a local "condition for integrability," and is not concerned with the behavior of $x(t)$ as $t\to\infty$. If the kernel $K(s, t)$ is of *finite reference* (that is, if for each s, $K(s, t)=0$ for all sufficiently great t), then $x \in \mathcal{K}$ implies existence of the K transform $y(s)$; but otherwise $x \in \mathcal{K}$ does not ordinarily imply existence of $y(s)$. For many transformations, the condition $x \in \mathcal{K}$ is precisely the condition that $x(t)$ be integrable over each finite interval $(0, h)$.

The class $\mathcal{K}$ is linear, that is, if $x_1, x_2 \in \mathcal{K}$ and c_1, c_2 are complex constants, then $c_1x_1+c_2x_2 \in \mathcal{K}$.

The conditions which we have imposed upon $\int$ and K are sufficient to make $\mathcal{K}$ an extensive class of functions. Indeed, some of the conditions were imposed to ensure that $\mathcal{K}$ contains the functions $x_0(t)$ which are described in the following lemma, and which we shall use later.

LEMMA 4.1. *Let* $0=h_1<h_2<\cdots$ *be a sequence for which* $\lim_{n\to\infty}h_n=\xi$ *may be either finite or* $+\infty$. *Let* $s_1, s_2, \cdots$ *be a sequence of positive numbers. Let* $\sigma_1, \sigma_2, \cdots$ *be a sequence of real numbers with* $|\sigma_n|\leqq 1$ *for each* n. *Let* $x_0(t)$ *be defined by the formulas*

$$x_0(t) = \sigma_n \operatorname{sgn} K(s_n, t), \qquad h_n \leqq t < h_{n+1}, \tag{4.11}$$

and, in case $\xi<\infty$, *by the additional formula*

$$x_0(t) = 0, \qquad t \geqq \xi. \tag{4.12}$$

Then $x_0 \in \mathfrak{K}$, $\mathfrak{R}x_0 \in \mathfrak{K}$, *and* $\mathfrak{I}x_0 \in \mathfrak{K}$.

It is easy to modify Lemma 4.1 and its proof to cover functions $x_0(t)$ defined analogously to (4.11) over intervals whose end points form a decreasing rather than an increasing sequence.

To prove that $\mathfrak{R}x_0 \in \mathfrak{K}$, let $s>0$ and $h>0$ be fixed. Our hypotheses on $\int$ and K imply existence of

$$\int_{h_n}^{h_{n+1}} K(s, t)\mathfrak{R} \operatorname{sgn} K(s_n, t)dt, \qquad n = 1, 2, \cdots. \tag{4.13}$$

Since σ_n is real, it follows that, if we set for simplicity $x_1(t)=\mathfrak{R}x_0(t)$, then

$$\int_{h_n}^{h_{n+1}} K(s, t)x_1(t)dt, \qquad n = 1, 2, \cdots, \tag{4.14}$$

exists. Therefore, for each $m=1, 2, \cdots$,

$$F_1(b, s) = \int_0^b K(s, t)x_1(t)dt \tag{4.15}$$

exists when $b=h_m$. If $h<\xi$, we can choose $h_m>h$ and conclude existence of $F_1(h, s)$. If $\xi=+\infty$, this implies that $x_1=\mathfrak{R}x_0 \in \mathfrak{K}$. If $\xi<\infty$, then existence of $F_1(b, s)$ for $b<\xi$, the fact that $K(s, t)x_1(t)$ is bounded over $0\leqq t\leqq \xi$ if $K(s, t) \in I(0, \xi)$ requires that $K(s, t)$ be bounded over $0\leqq t\leqq\xi$, and the inequality

$$\int_0^b |K(s, t)x_1(t)|\, dt \leqq \int_0^\xi |K(s, t)|\, dt, \qquad 0 < b < \xi, \tag{4.16}$$

imply by VII of §2 that $F_1(\xi, s)$ exists. If $h>\xi$, then existence of $F_1(\xi, s)$ and (4.12) imply existence of $F_1(h, s)$. Therefore $\mathfrak{R}x_0=x_1 \in \mathfrak{K}$. We can prove in the same way that $\mathfrak{I}x_0 \in \mathfrak{K}$, and linearity of $\mathfrak{K}$ then gives $x_0 \in \mathfrak{K}$.

5. **Regularity.** We recall that a matrix transformation A is said to be regular if each convergent sequence is summable A to the value to which it converges, that is, if existence of lim x_t implies existence of $y_0, y_1, \cdots$ and the equality lim $y_s = \lim x_t$. Necessary and sufficient conditions for regularity of A are, by the Silverman-Toeplitz theorem,

$$\sum_{t=0}^{\infty} | A_{st} | < M, \qquad s = 0, 1, 2, \cdots, \tag{5.01}$$

$$\lim_{s\to\infty} A_{st} = 0, \qquad t = 0, 1, 2, \cdots, \tag{5.02}$$

$$\lim_{s\to\infty} \sum_{t=0}^{\infty} A_{st} = 1, \tag{5.03}$$

M being a constant independent of s.

We use this theorem, and the theorems given later in this paragraph, merely for purposes of analogy; their truth is well known and will be demonstrated at the end of §10. The matrix A (or matrix transformation A) is *regular over a class* C of sequences if each convergent sequence $x \in C$ is summable A to the value to which it converges. The conditions (5.01) and (5.02) are necessary and sufficient to ensure that A be regular over the class of null sequences (sequences converging to 0). Also the conditions (5.01), (5.02), and (5.03) are necessary as well as sufficient for regularity of A (which may or may not be real) over the class of real sequences.

If we should call a kernel transformation K regular only when it has the property that each function $x(t)$ for which $\lim_{t\to\infty} x(t)$ exists has a transform $y(s)$ for which $\lim_{s\to\infty} y(s) = \lim_{t\to\infty} x(t)$, then it would turn out that no kernel transformation whatever involving a Riemann or Lebesgue integral could be regular. Consider, for example, the transformation

$$y(s) = \frac{1}{s} \int_0^s x(t)dt, \qquad s > 0, \tag{5.04}$$

in which the integral is that of Lebesgue. If $x(t) = 1/t^2$, then $\lim x(t) = 0$; but $y(s)$ does not exist (may be said to be $+\infty$) for each $s > 0$ and accordingly $\lim_{s\to\infty} y(s) = 0$ fails. Also, if $x(t)$ is a bounded function for which $\lim x(t) = 0$ but which is non-measurable over each finite interval, then again $y(s)$ fails to exist for each $s > 0$ and $\lim_{s\to\infty} y(s) = 0$ fails. However, if $x(t)$ belongs to the class of functions which are integrable over each finite interval and converge as $t\rightarrow\infty$, then it is well known and easy to show that $y(s)$ defined by (5.04) exists and that $\lim_{s\rightarrow\infty}y(s)=\lim_{t\rightarrow\infty}x(t)$.

These considerations imply that the following definition, which is analogous to one involving matrix transformations, will be useful. The transformation K is *regular over the class* C of functions x if each $x \varepsilon C$ for which $\lim x(t)$ exists is summable K to $\lim x(t)$. For example, (5.04) with Lebesgue integral is regular over the class C_1 of continuous functions and is also regular over the larger class C_2 of functions Lebesgue integrable over each finite interval $(0, h)$.

The largest class of functions over which a transformation K could possibly be regular is the class $\mathfrak{K}$ of §4. Accordingly, we define K to be *regular* if it is regular over the class $\mathfrak{K}$.

Regularity is only one of several properties in which we shall be interested. However, the conditions which characterize regular transformations K are so important in the theory of summability that it seems desirable to present them here and to discuss them briefly.

THEOREM 5.1. *In order that K be regular over $\mathfrak{K}$, it is necessary and sufficient that*

$$\int_0^\infty |K(s,t)|\,dt < \infty, \qquad s > 0, \tag{5.11}$$

$$\limsup_{s\rightarrow\infty}\int_0^\infty |K(s,t)|\,dt = M < \infty, \tag{5.12}$$

$$\lim_{s\rightarrow\infty}\int_0^h K(s,t)x(t)dt = 0, \qquad h > 0,\ x \varepsilon \mathfrak{K}, \tag{5.13}$$

$$\lim_{s\rightarrow\infty}\int_0^\infty K(s,t)dt = 1. \tag{5.14}$$

In the statement of this theorem, the phrase "K be regular over $\mathfrak{K}$" is used instead of the equivalent phrase "K be regular" in order to facilitate the statement of closely related theorems in §9. Theorem 5.1 will be proved in §9. Meanwhile we assume it and discuss the conditions involved.

The analogy between (5.01) on the one hand and (5.11) and (5.12) on the other hand seems satisfactory when we bear in mind that a sequence $\Sigma_0, \Sigma_1, \Sigma_2, \cdots$ of real numbers is bounded whenever $\lim\sup|\Sigma_n|<\infty$, but that a corresponding conclusion involving functions cannot be drawn. It is obvious either from the definition of regularity or from the conditions of Theorem 5.1 that, if $K_1(s, t)$ is a regular kernel, and $\phi(s)$ is any function of s which is defined for

$s>0$ and converges to 1 as $s\to\infty$ (say $\phi(s)=1+\psi(s)/s^2$ where $\psi(s)$ is a bounded non-measurable function which converges to 1 as $s\to 0$), then $K_2(s,t)=\phi(s)K_1(s,t)$ is also regular. In particular, if $K_1(s,t)$ is the simple kernel of (1.02), then both K_1 and K_2 are regular, and the equality

$$(5.15)\qquad \int_0^\infty |K_2(s,t)|\,dt = \left|\int_0^\infty K_2(s,t)dt\right| = |\phi(s)|$$

shows that the condition

$$(5.16)\qquad \underset{s>0}{\text{l.u.b.}} \int_0^\infty |K(s,t)|\,dt < \infty,$$

which is analogous to (5.01), is not necessary for regularity of K. Indeed, the choice of K_1 and ϕ suggested above shows that K can be regular even though

$$(5.17)\qquad \underset{s>0}{\text{l.u.b.}} \int_0^h |K(s,t)|\,dt = \infty, \qquad h>0.$$

The condition (5.14) is an exact analogue of (5.03). The condition (5.13) is by no means as attractive as the corresponding condition (5.02). However the real test of (5.13) comes when one seeks to determine whether a given kernel satisfies it. Actually it is often easier to determine whether a given kernel satisfies (5.13) than to determine whether it satisfies (5.12), and from this point of view we may accept (5.13). We do not in this paper make any attempt to present (5.13) in a different or more attractive form. Some consequences of the conditions for regularity are given by the following theorem.

THEOREM 5.2. *If K is regular, then*

$$(5.21)\qquad \lim_{s\to\infty} \int_0^h K(s,t)dt = 0, \qquad h>0,$$

$$(5.22)\qquad \lim_{s\to\infty} \int_h^\infty K(s,t)dt = 1, \qquad h\geqq 0,$$

and

$$(5.23)\qquad 1 \leqq \lim_{h\to\infty} \limsup_{s\to\infty} \int_h^\infty |K(s,t)|\,dt \leqq M < \infty.$$

The condition (5.21) is obtained by setting $x(t)=1$ in (5.13); and (5.22) is implied by (5.14) and (5.12). The inequality

$$(5.24)\qquad 1 \leqq \limsup_{s\to\infty} \int_h^\infty |K(s, t)|\, dt \leqq M, \qquad h \geqq 0,$$

is implied by (5.12) and (5.22). Since the middle term of (5.24) is a monotone decreasing function of h, (5.24) implies (5.23).

The number

$$(5.25)\qquad \Lambda = \lim_{h\to\infty} \limsup_{s\to\infty} \int_h^\infty |K(s, t)|\, dt$$

of condition (5.23) turns out to be significant in determining the properties which K has or fails to have. If Λ is defined by (5.25), then

$$(5.26)\qquad \Lambda = \limsup_{h, s\to\infty} \int_h^\infty |K(s, t)|\, dt;$$

and conversely if Λ is defined by (5.26), then (5.25) holds.

It can be shown by an example that (5.11), (5.12), (5.21), and (5.14) are not sufficient to ensure regularity of K.

The condition $\lim_{s\to\infty} A_{st} = 0$ being necessary for regularity of a matrix A, so also is the condition

$$(5.27)\qquad \lim_{s\to\infty} \sum_{t=0}^{h} |A_{st}| = 0, \qquad h = 0, 1, 2, \cdots .$$

To emphasize the fact that conditions analogous to these are not necessary for regularity of K, we give

THEOREM 5.3. *Neither of the conditions*

$$(5.31)\qquad \lim_{s\to\infty} K(s, t) = 0, \qquad t > 0,$$

$$(5.32)\qquad \lim_{s\to\infty} \int_0^h |K(s, t)|\, dt = 0, \qquad h > 0,$$

is necessary for regularity of K.

We prove this theorem by an example in which the integral is that of Lebesgue, and $x \in \mathcal{K}$ is equivalent to the condition that $x(t)$ be Lebesgue integrable over each finite interval $(0, h)$. If $x \in \mathcal{K}$ and $\limsup |x(t)| < \infty$, then

$$(5.33)\qquad \begin{aligned} y(s) = \int_0^s \frac{e^{ist}}{1+t^2}\, x(t)dt &+ \int_s^{s+1} x(t)dt \\ &+ \int_{s+1}^\infty \frac{e^{is(t-1)}}{1+(t-1)^2}\, x(t)dt \end{aligned}$$

exists for each $s \geqq 0$. For each $s \geqq 0$, $K(s, t)$ is $e^{ist}/(1+t^2)$ or 1 or $e^{is(t-1)}/[1+(t-1)^2]$ according as $0 \leqq t < s$ or $s \leqq t < s+1$ or $s+1 \leqq t$. Elementary integration gives, for each $s \geqq 0$,

$$(5.34) \qquad \int_0^\infty |K(s, t)| dt = \int_0^\infty \frac{1}{1+t^2} dt + \int_s^{s+1} 1 ds = \frac{\pi}{2} + 1.$$

By the well known Riemann-Lebesgue theorem

$$(5.35) \qquad \lim_{s\to\infty} \int_0^h K(s, t)x(t)dt = \lim_{s\to\infty} \int_0^h e^{ist}[x(t)/(1+t^2)]dt = 0$$

for $h > 0$, $x \in \mathfrak{K}$. Also,

$$\int_0^\infty K(s, t)dt = 1 + \int_0^\infty e^{ist}[1/(1+t^2)]dt,$$

and another application of the Riemann-Lebesgue theorem gives

$$(5.36) \qquad \lim_{s\to\infty} \int_0^\infty K(s, t)dt = 1.$$

It follows from Theorem 5.1 that the transformation (5.33) is regular. But there is no $t > 0$ for which $\lim_{s\to\infty} K(s, t)$ exists; and if $0 \leqq a < b$, then for all sufficiently great s

$$\int_a^b |K(s, t)| dt = \int_a^b \frac{1}{1+t^2} dt > 0,$$

so that

$$(5.37) \qquad \lim_{s\to\infty} \int_a^b |K(s, t)| dt$$

exists and is positive. This proves Theorem 5.3.

We give an example of a simple pair of regular transformations of finite reference to show that the conditions (5.31) and (5.32) are not significant in the theory of summability. Let K_1 and K_2 be defined by

$$K_1: \qquad y_1(s) = \int_0^1 \sigma(s)e^{ist}x(t)dt + \int_s^{s+1} x(t)dt,$$

$$K_2: \qquad y_2(s) = \int_s^{s+1} x(t)dt,$$

where $\sigma(s)$ is 0 or 1 according as $[s]$, the greatest integer less than or equal to s, is even or odd, and where the integrals are Lebesgue integrals. For K_1, the limits in (5.31) and (5.32) both fail to exist; for

K_2, both (5.31) and (5.32) hold. But $\mathfrak{K}_1 \equiv \mathfrak{K}_2$, each being the class of functions integrable over each finite interval $(0, h)$; and if $x \varepsilon \mathfrak{K}_1$, then by the Riemann-Lebesgue theorem,

$$\lim_{s\to\infty} | y_2(s) - y_1(s) | = 0.$$

Hence K_1 and K_2 are, as methods of summability, substantially identical.

In the next two sections, we give theorems which show that even very moderate hypotheses on a transformation K imply that (5.11) and (5.12) must hold.

6. **Application of K to functions $x \varepsilon \mathfrak{R}\mathfrak{K}_B(h)$.** Let $\mathfrak{K}_B(h)$ denote the class of functions $x(t)$ such that $x \varepsilon \mathfrak{K}$, $x(t)$ is bounded, and $x(t) = 0$ for all $t > h$. If C is a class of functions, we use $\mathfrak{R}C$ to denote the subclass of C consisting of the real functions in C. In several theorems of this and the next section, H denotes a real nonnegative constant which will be 0 in most of our applications.

THEOREM 6.1. *Let $h > 0$ and $H \geqq 0$ be fixed. If K is such that, for each $x \varepsilon \mathfrak{R}\mathfrak{K}_B(h)$,*

$$(6.11) \qquad y(s) = \int_0^\infty K(s, t)x(t)dt \equiv \int_0^h K(s, t)x(t)dt$$

exists for $s > H$ and has the property

$$(6.12) \qquad \underset{s>H}{\text{l.u.b.}} \, | y(s) | < \infty,$$

then there is a constant $M \equiv M(h) < \infty$ such that

$$(6.13) \qquad \underset{s>H}{\text{l.u.b.}} \int_0^h | K(s, t) | dt = M.$$

THEOREM 6.2. *Let $h > 0$ be fixed. If K is such that, for each $x \varepsilon \mathfrak{R}\mathfrak{K}_B(h)$,*

$$(6.21) \qquad y(s) = \int_0^\infty K(s, t)x(t)dt \equiv \int_0^h K(s, t)x(t)dt$$

exists for $s > s_0(x)$, where $s_0(x)$ is a real number for each $x \varepsilon \mathfrak{R}\mathfrak{K}_B(h)$, and has the property

$$(6.22) \qquad \limsup_{s\to\infty} | y(s) | < \infty,$$

then there is a constant $M \equiv M(h) < \infty$ such that

(6.23) $$\limsup_{s\to\infty} \int_0^h |K(s,t)|\,dt = M.$$

We first prove Theorem 6.1 by assuming that

(6.24) $$\underset{s>H}{\text{l.u.b.}} \int_0^h |K(s,t)|\,dt = +\infty,$$

and constructing a real function $x \varepsilon \mathcal{K}$ such that $|x(t)| \leqq 1$, $x(t)=0$ for $t>h$, and the transform $y(s)$ of x exists for $s>0$ and has the property l.u.b.$|y(s)| = +\infty$.

Let $a_0=0$, $b_0=h$. When a_n and b_n have been chosen such that

(6.25) $$\underset{s>H}{\text{l.u.b.}} \int_{a_n}^{b_n} |K(s,t)|\,dt = +\infty,$$

the properties of $\int$ given in §2 imply that at least one of the two definitions $[a_{n+1}=a_n;\ b_{n+1}=(a_n+b_n)/2]$ and $[a_{n+1}=(a_n+b_n)/2;\ b_{n+1}=b_n]$ will make (6.25) hold when n is replaced by $(n+1)$. Thus we obtain by induction a monotone increasing sequence $a_0, a_1, a_2, \cdots$ and a monotone decreasing sequence $b_0, b_1, b_2, \cdots$, such that $b_n-a_n=h/2^n$ and (6.25) holds for each $n=0, 1, 2, \cdots$. Let ξ be the common limit of the sequences a_n and b_n, so that

(6.26) $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \xi.$$

It is easy to show that at least one of the two functions of s

(6.27) $$\int_{a_n}^{\xi} |K(s,t)|\,dt, \qquad \int_{\xi}^{b_n} |K(s,t)|\,dt,$$

is unbounded for an infinite set of n and hence for each $n=0, 1, 2, \cdots$. Treatment of the alternative case being similar, we consider only the case that the first is unbounded for each $n=0, 1, 2, \cdots$. In this case

(6.28) $$F(s,c) = \int_c^{\xi} |K(s,t)|\,dt, \qquad s > H,$$

is an unbounded function of s for each c in the interval $0 \leqq c < \xi$.

Let $c_1=0$. Since $F(s, c_1)$ is unbounded, we can choose $s_1>H$ such that $F(s_1, c_1)>2+2^{-1}$. Since existence of (3.06) implies that $\lim_{c\to\xi} F(s_1, c)=0$, we can choose c_2 such that $c_1<c_2<\xi$, $\xi-c_2<2^{-2}$, and $F(s_1, c_2)<2^{-1}$. Next, choose $s_2>H$ such that $F(s_2, c_2)>2^2+2^{-2}$, and then choose c_3 such that $c_2<c_3<\xi$, $\xi-c_3<2^{-3}$, and $F(s_2, c_3)<2^{-2}$. Proceeding in this manner, we obtain a sequence $s_1, s_2, \cdots$ of values of $s>H$ and a sequence $c_1<c_2<c_3<\cdots$ of values of c such that $c_n\to\xi$, and for each $n=1, 2, \cdots$

$$(6.29)\qquad \int_{c_n}^{\xi} |K(s_n, t)|\,dt > 2^n + 2^{-n}, \qquad \int_{c_{n+1}}^{\xi} |K(s_n, t)|\,dt < 2^{-n},$$

and therefore also

$$(6.31)\qquad \int_{c_n}^{c_{n+1}} |K(s_n, t)|\,dt > 2^n.$$

We are now ready to define a function $x_0(t)$ in terms of which the desired function $x(t)$ will be determined. The function x_0 will be of the type described in Lemma 4.1, and accordingly $x_0 \varepsilon \mathcal{K}$. First, let

$$(6.32)\qquad x_0(t) = \operatorname{sgn} K(s_1, t), \qquad 0 = c_1 \leqq t < c_2.$$

Then, using (6.31) with $n=1$, we find that

$$(6.33)\qquad \left|\int_0^{c_{n+1}} K(s_n, t)x_0(t)dt\right| > 2^{n-1}$$

holds when $n=1$. When we have extended the definition of $x_0(t)$ over the interval $0\leqq t<c_{n+1}$ in such a way that (6.33) holds, we define $x_0(t)$ for $c_{n+1}\leqq t<c_{n+2}$ by the formula

$$(6.34)\qquad x_0(t) = \sigma_{n+1} \operatorname{sgn} K(s_{n+1}, t),$$

where σ_{n+1} is 0 if

$$(6.35)\qquad \left|\int_0^{c_{n+1}} K(s_{n+1}, t)x_0(t)dt\right| > 2^n$$

holds, and σ_{n+1} is 1 if (6.35) fails. If (6.35) holds, then $\sigma_{n+1}=0$, so that

$$(6.36)\qquad \left|\int_0^{c_{n+2}} K(s_{n+1}, t)x_0(t)dt\right| = \left|\int_0^{c_{n+1}} K(s_{n+1}, t)x_0(t)dt\right| > 2^n.$$

If (6.35) fails, then $\sigma_{n+1}=1$ and our inequalities give

$$\begin{aligned}
&\left|\int_0^{c_{n+2}} K(s_{n+1}, t)x_0(t)dt\right| \\
&\qquad \geqq \left|\int_{c_{n+1}}^{c_{n+2}} K(s_{n+1}, t)x_0(t)dt\right| - \left|\int_0^{c_{n+1}} K(s_{n+1}, t)x_0(t)dt\right| \\
&\qquad \geqq \int_{c_{n+1}}^{c_{n+2}} |K(s_{n+1}, t)|\,dt - 2^n \geqq 2^{n+1} - 2^n = 2^n.
\end{aligned}$$

Thus, in both of the two cases, we find that (6.33) holds when n is replaced by $n+1$.

This induction serves to define $x_0(t)$ over $0 \leqq t < \xi$; let $x_0(t)=0$ for $t \geqq \xi$. Then, since $x_0 \in \mathcal{K}_B(h)$, the transform $y_0(s)$ of x_0 exists for each $s>0$. For each $n=1, 2, \cdots$ we find

$$| y_0(s_n) | = \left| \int_0^{\xi} K(s_n, t) x_0(t) dt \right|$$

$$\geqq \left| \int_0^{c_{n+1}} K(s_n, t) x_0(t) dt \right| - \int_{c_{n+1}}^{\xi} | K(s_n, t) dt | > 2^{n-1} - 2^{-n}.$$

This implies that l.u.b.$| y_0(s) | = +\infty$. While $x_0 \in \mathcal{K}_B(h)$, it is not true that x_0 need be real unless K is real. Let $x_0(t)=x_1(t)+ix_2(t)$, where $x_1(t)$ and $x_2(t)$ are real. Then by Lemma 4.1, $x_1, x_2 \in \mathcal{K}$ and hence $x_1, x_2 \in \mathcal{K}_B(h)$. If we let $y_1(s)$ and $y_2(s)$ denote the transforms of x_1 and x_2, then

$$| y_1(s_n) + iy_2(s_n) | = | y_0(s_n) | > 2^{n-1} - 2^{-n}. \tag{6.37}$$

This implies that not both l.u.b. $y_1(s)$ and l.u.b. $y_2(s)$ can be finite. Accordingly, if $x(s)$ denotes a properly chosen one of the functions $x_1(s)$ and $x_2(s)$, then $x \in \mathcal{R}\mathcal{K}_B(h)$ and the transform y of x is unbounded. This proves Theorem 6.1.

Proof of Theorem 6.2 is practically identical with that of Theorem 6.1. Contradiction of the conclusion (6.23), which is weaker than (6.13), gives

$$\limsup_{s\to\infty} \int_0^h | K(s, t) | dt = +\infty, \tag{6.38}$$

which is stronger than (6.24) and enables us to choose the sequence $s_1, s_2, \cdots$ in such a way that $s_{n+1} > s_n + 1$ and hence $s_n \to \infty$ as $n \to \infty$. Then (6.37) would imply lim sup $| y(s) | = +\infty$ and thus complete the proof of Theorem 6.2.

7. **Application of K to functions $x \in \mathcal{K}_0$.** Let $\mathcal{K}_0$ denote the subclass of functions $x \in \mathcal{K}$ which are bounded over $0<t<\infty$ and converge to 0 as $t \to \infty$. It is clear that for each $h>0$ the class $\mathcal{K}_B(h)$ is a subclass of $\mathcal{K}_0$. The main theorems which we prove in this section are 7.4 and 7.5. We give first four simpler theorems which we shall need.

THEOREM 7.1. *If K is such that for each $x \in \mathcal{R}\mathcal{K}_0$*

$$\limsup_{h\to\infty} \left| \int_0^h K(s, t) x(t) dt \right| < \infty, \qquad s > H, \tag{7.11}$$

then

$$\int_0^\infty |K(s,t)|\,dt < \infty, \qquad s > H. \tag{7.12}$$

If (7.12) fails, then for some $s_0 > H$,

$$\lim_{h\to\infty} \int_0^h |K(s_0,t)|\,dt = \infty,$$

and we can choose a sequence $0 = h_1 < h_2 < \cdots$ of integers such that

$$\int_{h_n}^{h_{n+1}} |K(s_0,t)|\,dt \geqq n, \qquad n = 1, 2, \cdots.$$

If we set, for each $n = 1, 2, \cdots$,

$$x_0(t) = (1/n)\operatorname{sgn} K(s_0,t), \qquad h_n \leqq t < h_{n+1},$$

we find, using Lemma 4.1, that $x_0 \varepsilon \mathcal{K}_0$ and

$$\int_0^{h_{m+1}} K(s_0,t)x_0(t)dt = \sum_{n=1}^{m} \frac{1}{n}\int_{h_n}^{h_{n+1}} |K(s,t)|\,dt \geqq m, \qquad m = 1, 2, \cdots. \tag{7.13}$$

Hence

$$\limsup_{h\to\infty} \left| \int_0^h K(s_0,t)x_0(t)dt \right| = \infty. \tag{7.14}$$

Using again Lemma 4.1, we see that $x_1 = \mathfrak{R}x_0$ and $x_2 = \mathfrak{I}x_0$ are both members of $\mathfrak{R}\mathcal{K}_0$. It then follows from (7.14) that the statement obtained by replacing x_0 in (7.14) by one or the other of x_1 or x_2 must be true. This contradicts the hypothesis of Theorem 7.1, and Theorem 7.1 is accordingly proved. This proof has been so phrased that it serves also as a proof of the following theorem.

THEOREM 7.15. *If K is such that $K(s, t)$ is independent of t over each interval $n \leqq t < n+1$, $n = 0, 1, 2, \cdots$, and if*

$$\limsup_{n\to\infty} \left| \int_0^n K(s,t)x(t)dt \right| < \infty, \qquad s > H, \tag{7.16}$$

for each real function $x(t)$ which is constant over each interval $n \leqq t < n+1$ and converges to 0 as $t\to\infty$, then

$$\int_0^\infty |K(s,t)|\,dt < \infty, \qquad s > H. \tag{7.17}$$

THEOREM 7.2. *If K is such that for each $x \in \mathcal{RK}_0$*

$$\limsup_{h\to\infty}\left|\int_0^h K(s,t)x(t)dt\right| < \infty \tag{7.21}$$

for $s > s_0(x)$, where $s_0(x) < \infty$ is a real number for each $x \in \mathcal{RK}_0$, then a constant $H < \infty$ exists such that

$$\int_0^\infty |K(s,t)|\,dt < \infty, \qquad s > H. \tag{7.22}$$

This theorem would reduce essentially to Theorem 7.1 if it were assumed that a number $H < \infty$ exists such that $s_0(x) < H$ for all $x \in \mathcal{RK}_0$; but this is not assumed and accordingly Theorem 7.2 is essentially different from Theorem 7.1. To prove Theorem 7.2, we assume that the conclusion fails and obtain a contradiction of the hypothesis. Since

$$\int_0^h |K(s,t)|\,dt$$

exists for $h, s > 0$, failure of the conclusion implies existence of a sequence $0 < s_1 < s_2 < s_3 < \cdots$ such that $s_n \to \infty$ and

$$\int_0^\infty |K(s_n,t)|\,dt = \lim_{h\to\infty}\int_0^h |K(s_n,t)|\,dt = \infty, \quad n = 1, 2, \cdots. \tag{7.23}$$

Let $h_0 = 0$ and choose h_1 such that

$$\int_{h_0}^{h_1} |K(s_1,t)|\,dt > 2.$$

Next, choose $h_2 > h_1 + 1$ such that

$$\int_{h_1}^{h_2} |K(s_p,t)|\,dt > 2^2, \qquad p = 1, 2.$$

When the $h_0, h_1, \cdots, h_{n-1}$ are determined, choose $h_n > h_{n-1} + 1$ such that

$$\int_{h_{n-1}}^{h_n} |K(s_p,t)|\,dt > 2^n, \qquad p = 1, 2, \cdots, n. \tag{7.24}$$

Let $\alpha_1, \alpha_2, \cdots$ denote in order the integers in the sequence

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \cdots, 1, 2, \cdots, m, \cdots.$$

Since $\alpha_n \leqq n$ for each $n = 1, 2, \cdots$, it follows from (7.24) that

$$\int_{h_{n-1}}^{h_n} |K(s_{\alpha_n}, t)| \, dt > 2^n, \qquad n = 1, 2, \cdots. \tag{7.25}$$

Let $x_0(t)$ be defined over $0 \leqq t < \infty$ by the formulas

$$x_0(t) = n^{-1} \operatorname{sgn} K(s_{\alpha_n}, t), \qquad h_{n-1} \leqq t < h_n. \tag{7.26}$$

Then, by Theorem 4.1, $x_0 \varepsilon \mathcal{K}$, and (7.26) implies further that $x_0 \varepsilon \mathcal{K}_0$. Now let q be a positive integer, and let

$$F_q(h) = \int_0^h K(s_q, t) x_0(t) dt. \tag{7.27}$$

For each of the infinite set of values of n for which $\alpha_n = q$, we can use (7.27), (7.26) and (7.25) to obtain

$$F_q(h_n) - F_q(h_{n-1}) = \int_{h_{n-1}}^{h_n} K(s_{\alpha_n}, t) x_0(t) dt$$

$$= \frac{1}{n} \int_{h_{n-1}}^{h_n} |K(s_{\alpha_n}, t)| \, dt > n^{-1} 2^n.$$

Therefore,

$$\limsup_{h \to \infty} \left| \int_0^h K(s_q, t) x_0(t) dt \right| = \infty \tag{7.28}$$

for each $q = 1, 2, 3, \cdots$. If we let $x_1 = \mathfrak{R} x_0$, $x_2 = \mathfrak{J} x_0$, then $x_1, x_2 \varepsilon \mathcal{K}$ by Theorem 4.1 and hence $x_1, x_2 \varepsilon \mathfrak{R}\mathcal{K}_0$. Moreover, if x_0 is replaced in (7.28) by one or the other (perhaps either) of the functions x_1 and x_2, then (7.28) must hold for an infinite set of values of q. Since $s_q \to \infty$, this contradicts the hypothesis of Theorem 7.2, and the proof of Theorem 7.2 is complete.

THEOREM 7.3. *Let $H \geqq 0$ be fixed. If*

$$\int_0^\infty |K(s, t)| \, dt < \infty, \qquad s > H, \tag{7.31}$$

then

$$y(s) = \int_0^\infty K(s, t) x(t) dt, \qquad s > H, \tag{7.32}$$

exists for each $x \varepsilon \mathcal{K}$ such that $\limsup |x(t)| < \infty$.

Let $x \varepsilon \mathcal{K}$, $\limsup |x(t)| < \infty$, and choose C and $t_0 > 0$ such that $|x(t)| < C$ when $t > t_0$. Let $s > H$ be fixed. Then, since $x \varepsilon \mathcal{K}$, it follows that $F(h) = \int_0^h K(s, t) x(t) dt$ exists for each $h > 0$, and if $q > p > t_0$ then

$$|F(q) - F(p)| = \left| \int_p^q K(s, t)x(t)dt \right| \leqq C \int_p^q |K(s, t)| dt.$$

This inequality and the Cauchy criterion for convergence imply existence of $y(s) = \lim F(h)$, and Theorem 7.3 is proved.

THEOREM 7.4. *Let $H \geqq 0$ be fixed. If K is such that for each $x \in \mathfrak{R}\mathfrak{K}_0$*

$$(7.41) \qquad y(s) = \int_0^\infty K(s, t)x(t)dt, \qquad s > H,$$

exists and has the property

$$(7.42) \qquad \underset{s>H}{\text{l.u.b.}} |y(s)| < \infty,$$

then there is a constant $M < \infty$ such that

$$(7.43) \qquad \underset{s>H}{\text{l.u.b.}} \int_0^\infty |K(s, t)| dt = M.$$

THEOREM 7.5. *If K is such that, for each $x \in \mathfrak{R}\mathfrak{K}_0$,*

$$(7.51) \qquad y(s) = \int_0^\infty K(s, t)x(t)dt$$

exists for $s \geqq s_0(x)$, where $s_0(x) < \infty$ is a real number for each $x \in \mathfrak{R}\mathfrak{K}_0$, and has the property

$$(7.52) \qquad \limsup_{s\to\infty} |y(s)| < \infty,$$

then there is a constant $M < \infty$ such that

$$(7.53) \qquad \limsup_{s\to\infty} \int_0^\infty |K(s, t)| dt = M.$$

We first prove Theorem 7.4. The hypothesis and Theorem 7.1 imply that

$$(7.61) \qquad \int_0^\infty |K(s, t)| dt < \infty, \qquad s > H.$$

Also, since $\mathfrak{R}\mathfrak{K}_B(h)$ is a subclass of $\mathfrak{R}\mathfrak{K}_0$, it follows from our hypothesis and Theorem 6.1 that for each $h > 0$ there is a constant $M(h)$ such that

$$\text{(7.62)} \qquad \underset{s>H}{\text{l.u.b.}} \int_0^h |K(s,t)|\,dt = M(h).$$

Assuming that (7.43) fails, that is, that

$$\text{(7.63)} \qquad \underset{s>H}{\text{l.u.b.}} \int_0^\infty |K(s,t)|\,dt = \infty,$$

we construct a function $x \varepsilon \mathcal{R}\mathcal{K}_0$ for which (7.42) fails. From (7.62) and (7.63), it follows that

$$\text{(7.64)} \qquad \underset{s>H}{\text{l.u.b.}} \int_h^\infty |K(s,t)|\,dt = \infty, \qquad h \geqq 0.$$

Let $h_0=0$ and $M(h_0)=0$. It follows from (7.64) that we can choose $s_1>H$ such that the inequality

$$\text{(7.65)} \qquad \int_{h_{n-1}}^\infty |K(s_n,t)|\,dt > 4^n[2+M(h_{n-1})]$$

will hold when $n=1$, and then choose h_1 so that $h_1>h_0+1$ and

$$\text{(7.66)} \qquad \begin{aligned} &\int_{h_{n-1}}^{h_n} |K(s_n,t)|\,dt > 4^n[2+M(h_{n-1})], \\ &\int_{h_n}^\infty |K(s_n,t)|\,dt < 2^{-n} \end{aligned}$$

hold when $n=1$. We continue in this way to obtain sequences $s_1, s_2, \cdots$ and $h_1<h_2<\cdots$, such that $s_n>H$, $h_{n+1}>h_n+1$ and (7.66) holds for each $n=1, 2, 3, \cdots$. Now let $x_0(t)$ be defined by

$$\text{(7.67)} \qquad x_0(t) = 2^{-n} \operatorname{sgn} K(s_n,t), \qquad h_{n-1} \leqq t < h_n,$$

in which n takes on values $1, 2, \cdots$. Then $x_0 \varepsilon \mathcal{K}$ by Lemma 4.1; and obviously $|x_0(t)| \leqq 1$ and $x_0(t)\to 0$, so that $x_0 \varepsilon \mathcal{K}_0$. Hence (7.61) and Theorem 7.3 imply that the transform $y_0(s)$ of $x_0(s)$ exists. The inequalities

$$\left|\int_{h_{n-1}}^{h_n} K(s_n,t)x_0(t)dt\right| = 2^{-n}\int_{h_{n-1}}^{h_n} |K(s_n,t)|\,dt > 2^n[2+M(h_{n-1})],$$

$$\left|\int_0^{h_{n-1}} K(s_n,t)x_0(t)dt\right| \leqq \int_0^{h_{n-1}} |K(s_n,t)|\,dt \leqq M(h_{n-1}),$$

$$\left|\int_{h_n}^\infty K(s_n,t)x_0(t)dt\right| \leqq \int_{h_n}^\infty |K(s_n,t)|\,dt \leqq 2^{-n}$$

imply that

$$| y_0(s_n) | > 2^n[2 + M(h_{n-1})] - M(h_{n-1}) - 2^{-n} > 2^n,$$

and hence

$$(7.68) \qquad \lim_{n\to\infty} | y_0(s_n) | = \infty .$$

Thus y_0 does not have a bounded transform. It follows that at least one of the two real functions $\mathfrak{R}x_0$ and $\mathfrak{I}x_0$ (which are both members of $\mathfrak{R}\mathcal{K}_0$ and have transforms) must fail to have a bounded transform and Theorem 7.4 is proved.

The proof of Theorem 7.5 is the same as that of Theorem 7.4 except that Theorem 7.2 is used to obtain (7.61), and that in (7.62), (7.63), and (7.64), "l.u.b.$_{s>H}$" is replaced by "lim sup$_{s\to\infty}$" so that we can choose the sequence $s_1, s_2, \cdots$ such that $s_{n+1} > s_n + 1$ and obtain a contradiction of (7.52).

8. **Application of** K **to functions** $x \in \mathcal{K}_B$. Let $\mathcal{K}_B$ denote the class of functions $x(t) \in \mathcal{K}$ which are bounded over $0 < t < \infty$. In this section, we give two closely related theorems.

THEOREM 8.1. *If K is such that*

$$(8.11) \qquad \int_0^\infty | K(s, t) | \, dt < \infty , \qquad s > 0,$$

and

$$(8.12) \qquad \lim_{h\to\infty} \operatorname*{l.u.b.}_{s>0} \int_h^\infty | K(s, t) | \, dt = \Lambda,$$

then (whether Λ is finite or $+\infty$) there is a function $x \in \mathcal{K}_B$ such that

$$(8.13) \qquad | x(t) | \leqq 1, \qquad t \geqq 0,$$

and the transform $y(s)$ of $x(t)$ exists for $s > 0$ and has the property

$$(8.14) \qquad \operatorname*{l.u.b.}_{u,v>0} | y(u) - y(v) | \geqq 2\Lambda.$$

Moreover if $K(s, t)$ is real, the function $x(t)$ may be taken real.

THEOREM 8.2. *If K is such that*

$$(8.21) \qquad \int_0^\infty | K(s, t) | \, dt < \infty , \qquad s > 0,$$

and

$$(8.22)\qquad \lim_{h\to\infty}\limsup_{s\to\infty}\int_h^\infty |K(s,t)|\,dt=\Lambda,$$

then (where Λ *is finite or* $+\infty$*) there is a function* $x \in \mathcal{K}_B$ *such that*

$$(8.23)\qquad |x(t)|\leqq 1, \qquad t\geqq 0,$$

and the transform $y(s)$ *of* $x(t)$ *exists and has the property*

$$(8.24)\qquad \limsup_{u,v\to\infty}|y(u)-y(v)|\geqq 2\Lambda.$$

Moreover if $K(s, t)$ *is real, the function* $x(t)$ *may be taken real.*

We first prove Theorem 8.1. The case $\Lambda=0$ is trivial. If $\Lambda=+\infty$, the left member of (7.43), with $H=0$, will be $+\infty$; hence, by Theorem 7.4, $x \in \mathcal{R}\mathcal{K}_0$ exists such that (7.42) fails. But (8.11) implies that the transform $y(s)$ of $x(t)$ exists; hence l.u.b. $|y(s)|=+\infty$ and (8.14) follows. We consider now the remaining case in which $0<\Lambda<\infty$.

The hypothesis (8.12) implies that we can choose $h_1>0$ such that

$$(8.31)\qquad \Lambda-\frac{1}{n}<\underset{s>0}{\text{l.u.b.}}\int_{h_n}^\infty |K(s,t)|\,dt<\Lambda+\frac{1}{n}$$

will hold when $n=1$, and then choose $s_1>0$ such that

$$(8.32)\qquad \Lambda-\frac{1}{n}<\int_{h_n}^\infty |K(s_n,t)|\,dt<\Lambda+\frac{1}{n}$$

holds when $n=1$. Next we can choose $h_2>h_1+1$ such that

$$(8.33)\qquad \int_{h_{n+1}}^\infty |K(s_n,t)|\,dt<\frac{1}{n}$$

holds when $n=1$, and (8.31) holds when $n=2$, and then choose $s_2>0$ such that (8.32) holds when $n=2$. We can continue by induction to define sequences h_n and s_n such that $h_{n+1}>h_n+1$, $s_n>0$, and (8.31), (8.32), and (8.33) hold for each $n=1, 2, \cdots$. From (8.32) and (8.33) we obtain

$$(8.34)\qquad \Lambda-\frac{2}{n}<\int_{h_n}^{h_{n+1}} |K(s_n,t)|\,dt<\Lambda+\frac{2}{n}, \qquad n=1,2,\cdots.$$

Let $x(t)=0$ when $0\leqq t<h_1$; and for each $n=1, 2, \cdots$ let

$$(8.35)\qquad x(t)=\sigma_n \operatorname{sgn} K(s_n,t), \qquad h_n\leqq t<h_{n+1},$$

where each σ_n is one of the numbers $+1$ or -1 to be determined presently. Then however we determine the σ_n, $|x(t)| \leqq 1$; $x \, \varepsilon \, \mathcal{K}$ by Lemma 4.1; and our hypothesis and Theorem 7.3 imply that the transform $y(s)$ of $x(t)$ exists. For each $n=1, 2, \cdots$ we find

$$(8.36) \qquad y(s_n) = A_n + \sigma_n \Lambda + R_n,$$

where

$$(8.37) \qquad A_n = \int_0^{h_n} K(s_n, t)x(t)dt = \int_{h_1}^{h_n} K(s_n, t)x(t)dt,$$

and

$$(8.38) \qquad R_n = \sigma_n \left[\int_{h_n}^{h_{n+1}} | K(s_n, t) | dt - \Lambda \right] + \int_{h_{n+1}}^{\infty} K(s_n, t)x(t)dt.$$

Using (8.31), we find

$$(8.39) \qquad | A_n | \leqq \int_{h_1}^{\infty} | K(s_n, t) | dt < \Lambda + 1,$$

and using (8.34) and (8.33) we find

$$(8.41) \qquad \lim_{n\to\infty} R_n = 0.$$

If we set $B_n = \mathcal{R} A_n$ and $C_n = \mathcal{R} R_n$, then

$$(8.42) \qquad \mathcal{R} y(s_n) = B_n + \sigma_n \Lambda + C_n.$$

If we set

$$(8.43) \qquad X_n = B_n + \sigma_n \Lambda,$$

then (8.41), (8.42), (8.43), and the definitions of B_n and C_n imply that

$$(8.44) \qquad \begin{aligned} \operatorname*{l.u.b.}_{u,v>0} | y(u) - y(v) | &\geqq \limsup_{m,n\to\infty} | y(s_m) - y(s_n) | \\ &\geqq \limsup_{m,n\to\infty} | \mathcal{R} y(s_m) - \mathcal{R} y(s_n) | \\ &= \limsup_{m,n\to\infty} | X_m - X_n |. \end{aligned}$$

Accordingly it is sufficient to determine the $\sigma_n = \pm 1$ so that

$$(8.45) \qquad \limsup_{m,n\to\infty} | X_m - X_n | \geqq 2\Lambda.$$

The definition of A_n shows that $A_1 = 0$ and hence $B_1 = \mathcal{R} A_1 = 0$, and that for each $n = 1, 2, \cdots$ the constants $\sigma_1, \sigma_2, \cdots, \sigma_n$ determine A_{n+1} and hence $B_{n+1} = \mathcal{R} A_{n+1}$. Moreover (8.39) and the definition

$B_n = \Re A_n$ imply that, however the σ_n are defined, the inequalities

$$(8.46) \qquad -(\Lambda+1) < B_n < \Lambda + 1, \qquad n = 1, 2, \cdots,$$

must hold.

We now indicate, by giving one step, how it is possible to define the σ_n by induction to obtain (8.45). Suppose σ_n has been defined for $n < n_p$ and accordingly B_n is determined for $n \leqq n_p$, p and n_p being positive integers. To simplify typography, let $q = n_p$. Let the interval $-(\Lambda+1) \leqq t < (\Lambda+1)$ in which all points B_n must lie be divided into p equal subintervals $I_1, I_2, \cdots, I_p$, each closed on the left and open on the right. Let $I^{(1)}$ denote the subinterval containing B_q. Let $\sigma_q = 1$, and let $I^{(2)}$ benote the subinterval containing B_{q+1}. If $I^{(2)} \neq I^{(1)}$, let $\sigma_{q+1} = 1$, but if $I^{(2)} = I^{(1)}$, let $\sigma_{q+1} = -1$; and then let $I^{(3)}$ denote the subinterval containing B_{q+2}. If $I^{(3)}$ differs from both $I^{(1)}$ and $I^{(2)}$, let $\sigma_{q+2} = 1$; otherwise let $\sigma_{q+2} = -1$. We continue in this manner to define σ_n for $n_p \leqq n < n_{p+1}$ where $n_{p+1} = n_p + p + 1$. Since the number $(p+1)$ of points B_n is greater than the number p of subintervals, there must be two indices α and β with $p_n \leqq \alpha < \beta < p_{n+1}$ such that

$$(8.47) \qquad |B_\alpha - B_\beta| < 2(\Lambda+1)/p,$$

and $\sigma_\alpha = +1$, $\sigma_\beta = -1$, so that

$$(8.48) \qquad X_\alpha = B_\alpha + \Lambda, \qquad X_\beta = B_\beta - \Lambda.$$

The two inequalities (8.47) and (8.48) imply

$$(8.49) \qquad X_\alpha - X_\beta > 2\Lambda - 2(\Lambda+1)/p.$$

The indices α and β in (8.49) depend upon p, and as p becomes infinite, so also do α and β. Hence (8.49) implies (8.45) and therefore (8.14). Finally if $K(s, t)$ is real, the function $x(t)$ which we constructed is real and Theorem 8.1 is proved.

From (8.44) and (8.45), we obtain

$$(8.51) \qquad \underset{u,v>0}{\text{l.u.b.}} \left| \Re y(u) - \Re y(v) \right| \geqq 2\Lambda;$$

hence we have proved the stronger theorem obtained by replacing (8.14) by (8.51) in Theorem 8.1. However this extension follows from Theorem 8.1 itself. For if $x(t)$ is a function of the required type whose transform satisfies (8.13), then we can choose a real angle θ such that $x(t)e^{i\theta}$ will be a function of the required type whose transform satisfies (8.51).

We turn now to proof of Theorem 8.2. The case $\Lambda = 0$ is trivial, and the case $\Lambda = \infty$ is covered by Theorem 7.5. In case $0 < \Lambda < \infty$, the hypothesis (8.22) enables us to choose a number $N>0$ such that

$$\int_h^\infty |K(s,t)|\,dt < \Lambda + 1, \qquad h, s \geqq N. \tag{8.52}$$

We can then choose sequences $N<h_1<h_2<\cdots$ and $N<s_1<s_2<\cdots$ such that $s_{n+1}>s_n+1$, $h_{n+1}>h_n+1$, the estimate obtained by replacing "l.u.b." by "lim sup" in (8.31) holds, and (8.32) and (8.33) hold. The proof is then precisely like that of Theorem 8.1 except that (8.39) is implied by (8.52) rather than the analogue of (8.22), and that "$\text{l.u.b.}_{u,v>0}$" is replaced by "$\limsup_{u,v\to\infty}$" in (8.44). We thus obtain (8.24) and Theorem 8.2 is proved.

THEOREM 8.6. *If C is a nonnegative constant, then a necessary and sufficient condition that*

$$\underset{s>0}{\text{l.u.b.}}\ |y(s)| \leqq C\ \underset{s>0}{\text{l.u.b.}}\ |x(s)|, \qquad x \in \mathfrak{K}, \tag{8.61}$$

is that

$$\underset{s>0}{\text{l.u.b.}} \int_0^\infty |K(s,t)|\,dt \leqq C.$$

Necessity is obtained by consideration of the different functions $x(t)=\text{sgn } K(s,t)$ obtained by giving different values to s, and sufficiency is easily established.

9. **Conditions that K be conservative, conservative for null sequences, multiplicative, regular, regular for null sequences, coercive, and null.** A transformation K is called *conservative* (convergence-preserving) *over a class C* of functions if $x \in C$ and existence of $\lim x(t)$ imply existence of $y(s)$ for $s>0$ and of $\lim y(s)$. Equality of $\lim y(s)$ and $\lim x(t)$ is neither required nor prohibited by this definition. If K is conservative over $\mathfrak{K}$, then K is called *conservative*. In the next theorem, we use the phrase "conservative over $\mathfrak{K}$" rather than "conservative" in order to facilitate statements of closely related theorems.

THEOREM 9.1. *Necessary and sufficient conditions that K be conservative over $\mathfrak{K}$ are*

$$\int_0^\infty |K(s,t)|\,dt < \infty, \qquad s>0, \tag{9.11}$$

$$\limsup_{s\to\infty} \int_0^\infty |K(s,t)|\,dt = M < \infty, \tag{9.12}$$

$$\lim_{s\to\infty}\int_0^h K(s, t)x(t)dt = L_h(x), \qquad h > 0,\ x \in \mathcal{K}, \tag{9.13}$$

$$\lim_{s\to\infty}\int_0^\infty K(s, t)dt = \rho, \tag{9.14}$$

where M and ρ are constants, and $L_h(x)$ is a constant for each choice of $h>0$, $x \in \mathcal{K}$.

Necessity of (9.11) and (9.12) follow respectively from Theorems 7.1 and 7.5. To prove necessity of (9.13), let $x \in \mathcal{K}$, let $h>0$ and let $x_0(t)=x(t)$ or 0 according as $0\leqq t\leqq h$ or $t>h$. Since $x_0 \in \mathcal{K}$ and $x_0(t)\to 0$, our hypotheses imply that the transform

$$y_0(s) = \int_0^\infty K(s,t)x_0(t)dt = \int_0^h K(s, t)x(t)dt \tag{9.15}$$

must converge to some limit [which we may denote by $L_h(x)$] as $s\to\infty$. Necessity of (9.14) results from the fact that the transform of the function $x(t)\equiv 1$ must converge as $s\to\infty$.

To prove sufficiency, let $x \in \mathcal{K}$ be such that $\lim_{t\to\infty}x(t)=\lambda$ exists. Let $\epsilon>0$. Choose $h>0$ such that

$$|x(t) - \lambda| < \epsilon, \qquad t \geqq h.$$

Then Theorem 7.3 implies that $y(s)$ exists for $s>0$, and we can use the estimate

$$\begin{aligned} |y(u) - y(v)| \leqq &\left|\int_0^h K(u, t)[x(t) - \lambda]dt - \int_0^h K(v, t)[x(t) - \lambda]dt\right| \\ &+ \int_h^\infty |K(u, t)||x(t) - \lambda| dt \\ &+ \int_h^\infty |K(v, t)||x(t) - \lambda| dt \\ &+ |\lambda|\left|\int_0^\infty K(u, t)dt - \int_0^\infty K(v, t)dt\right| \end{aligned}$$

and our hypotheses to obtain $\limsup_{u,v\to\infty}|y(u)-y(v)|\leqq 2\epsilon M$. Therefore $\lim_{u,v\to\infty}|y(u)-y(v)|=0$, and the Cauchy criterion for convergence implies existence of $\lim_{s\to\infty}y(s)$.

The theorems used in the proof of Theorem 9.1 are sufficiently general to enable us to replace $\mathcal{K}$ successively by $\mathcal{RK}$, $\mathcal{K}_B$ and $\mathcal{RK}_B$ in the proof of Theorem 9.1 to obtain

THEOREM 9.16. *Theorem* 9.1 *remains true when* $\mathfrak{K}$ *is replaced by* $\mathfrak{R}\mathfrak{K}$ *or by* $\mathfrak{K}_B$ *or by* $\mathfrak{R}\mathfrak{K}_B$ *in its statement.*

THEOREM 9.17. *The conditions* (9.11), (9.12), *and* (9.13) *are necessary and sufficient that K be conservative over the class of null functions in* $\mathfrak{K}$ *(that is, functions* $x \in \mathfrak{K}$ *for which* $\lim x(t)=0$).

Proof of Theorem 9.17 is the same as that of Theorem 9.1 except that necessity of (9.14) need not be proved and that the condition (9.14) is not needed to establish existence of $\lim y(s)$ when $\lim x(t)=\lambda=0$. Replacing $\mathfrak{K}$ by $\mathfrak{R}\mathfrak{K}$ or $\mathfrak{K}_0$ or $\mathfrak{R}\mathfrak{K}_0$ gives characterizations of transformations conservative over the classes of real null functions in $\mathfrak{K}$ or bounded null functions in $\mathfrak{K}$ or real bounded null functions in $\mathfrak{K}$.

If K is conservative over C and such that for convergent functions $x \in C$ the value of $\lim y(s)$ is independent of the value of $x(t)$ on each interval $0 \leqq t \leqq h$, then K is called *multiplicative over C*. If K is multiplicative over $\mathfrak{K}$, then K is *multiplicative*.

THEOREM 9.2. *Necessary and sufficient conditions that K be multiplicative over* $\mathfrak{K}$ *are*

$$\int_0^\infty |K(s,t)|\,dt < \infty, \qquad s>0, \tag{9.21}$$

$$\limsup_{s\to\infty} \int_0^\infty |K(s,t)|\,dt = M < \infty, \tag{9.22}$$

$$\lim_{h\to\infty} \int_0^h K(s,t)x(t)dt = 0, \qquad h>0,\ x \in \mathfrak{K}, \tag{9.23}$$

$$\lim_{s\to\infty} \int_0^\infty K(s,t)dt = \rho. \tag{9.24}$$

Moreover if K is multiplicative, $x \in \mathfrak{K}$, *and* $\lim x(t)=\lambda$, *then x is summable K to* $\rho\lambda$.

The number ρ given by (9.24) is called the *multiplier* of the transformation. Necessity of (9.21), (9.22), and (9.24) follows from Theorem 9.1. To prove necessity of (9.23), let $x \in \mathfrak{K}$ and $h>0$ be fixed. Let $x_1(t)=x(t)$ or 0 according as $0\leqq t\leqq h$ or $t>h$, and let $x_2(t)\equiv 0$. Since K is multiplicative, the transforms $y_1(s)$ and $y_2(s)$ of x_1 and x_2 must have the same limit; hence

$$y_1(s) - y_2(s) = \int_0^h K(s,t)x(t)dt$$

must converge to 0 as $s\rightarrow\infty$ and proof of necessity is complete. To prove sufficiency, let $x(t)\ \varepsilon\ \mathcal{K}$ and let $\lim_{t\rightarrow\infty}x(t)=\lambda$. Let $\epsilon>0$. If $h>0$ is such that $|x(t)-\lambda|<\epsilon$ for $t\geqq h$, then the equality

$$y(s) - \rho\lambda = A(s) + B(s) + \int_h^\infty K(s, t)[x(t) - \lambda]dt,$$

where

$$A(s) = \lambda\left[\int_0^\infty K(s, t)dt - \rho\right], \qquad B(s) = \int_0^h K(s, t)[x(t) - \lambda]dt,$$

implies

$$| y(s) - \rho\lambda | \leqq | A(s) | + | B(s) | + \epsilon\int_h^\infty | K(s, t) | dt,$$

and we can use our hypotheses to obtain $\limsup_{s\rightarrow\infty}|y(s)-\rho\lambda|\leqq\epsilon M$. Therefore $\lim_{s\rightarrow\infty}y(s)=\rho\lambda$, that is, $x(t)$ is summable to $\rho\lambda$. Since $\rho\lambda$ is independent of the values of $x(t)$ over each finite interval $(0, h)$, the theorem is proved. Replacing $\mathcal{K}$ by $\mathcal{RK}$, $\mathcal{K}_B$, $\mathcal{RK}_B$ in this proof gives

THEOREM 9.25. *Theorem* 9.2 *remains true when* $\mathcal{K}$ *is replaced by* $\mathcal{RK}$ *or by* $\mathcal{K}_B$ *or by* $\mathcal{RK}_B$ *in its statement.*

We now prove Theorem 5.1 and related theorems. Let K be regular over C, where C is one of the classes $\mathcal{K}$ or $\mathcal{RK}$ or $\mathcal{K}_B$ or $\mathcal{RK}_B$. Then the definitions of regular and multiplicative transformations imply that K is multiplicative over C. Hence by Theorems 9.2 and 9.25 the function $x_0(t)\equiv 1$ is summable K to ρ. Regularity of K over C implies that $\rho=\lim x_0(t)=1$. On the other hand Theorems 9.2 and 9.25 imply that if K is multplicative over C and $\rho=1$, then K is regular over C. Thus the class of transformations regular over C is identical with the class of transformations multiplicative over C with multiplier $\rho=1$. This fact and Theorems 9.2 and 9.25 prove Theorem 5.1 and the following one.

THEOREM 9.3. *Theorem* 5.1 *remains true when* $\mathcal{K}$ *is replaced by* $\mathcal{RK}$ (the class of real functions in $\mathcal{K}$) *or by* $\mathcal{K}_B$ (the class of bounded functions in $\mathcal{K}$) *or by* $\mathcal{RK}_B$ (the class of real bounded functions in $\mathcal{K}$) *in its statement.*

Methods of proof already used in this section suffice to prove the following two theorems.

THEOREM 9.4. *In order that K be regular over the class of null functions in* $\mathcal{K}$, *it is necessary and sufficient that*

$$\int_0^\infty |K(s,t)|\,dt < \infty, \qquad s > 0, \tag{9.41}$$

$$\limsup_{s\to\infty} \int_0^\infty |K(s,t)|\,dt < \infty, \tag{9.42}$$

$$\lim_{s\to\infty} \int_0^h K(s,t)x(t)dt = 0, \qquad h > 0,\ x \in \mathcal{K}. \tag{9.43}$$

THEOREM 9.44. *Theorem* 9.4 *remains true when* $\mathcal{K}$ *is replaced by* $\mathcal{RK}$ *or* $\mathcal{K}_B$ *or* $\mathcal{RK}_B$ *in its statement.*

A transformation K is called *coercive* if each $x \in \mathcal{K}_B$ (that is, each bounded function $x \in \mathcal{K}$) is summable K.

THEOREM 9.5. *Necessary and sufficient conditions that K be coercive are*

$$\int_0^\infty |K(s,t)|\,dt < \infty, \qquad s > 0, \tag{9.51}$$

$$\lim_{s\to\infty} \int_0^h K(s,t)x(t)dt = L_h(x), \qquad h > 0,\ x \in \mathcal{K}_B, \tag{9.52}$$

$$\lim_{h\to\infty} \limsup_{s\to\infty} \int_h^\infty |K(s,t)|\,dt = 0. \tag{9.53}$$

Necessity of (9.51) and (9.52) follows from Theorem 9.15 and the fact that each coercive transformation is conservative over $\mathcal{K}_B$. Necessity of (9.53) follows from Theorem 8.2; for, if the left member of (9.53) is $\Lambda > 0$, then there is a function $x \in \mathcal{K}_B$ for which (8.24) holds, and therefore x is not summable K. To establish sufficiency, let $x \in \mathcal{K}$ be such that l.u.b. $|x(t)| < \lambda < \infty$. Let $\epsilon > 0$. Choose $h > 0$ and $s_0 > 0$ such that

$$\int_h^\infty |K(s,t)|\,dt < \epsilon/2\lambda, \qquad s \geqq s_0.$$

The hypothesis (9.51) guarantees that the transform $y(s)$ of $x(t)$ exists. Hence

$$|y(u) - y(v)| \leqq |F(u) - F(v)| + |G(u)| + |G(v)|, \qquad u, v > 0,$$

where

$$F(s) = \int_0^h K(s,t)x(t)dt, \qquad G(s) = \int_h^\infty K(s,t)x(t)dt.$$

Our hypotheses and inequalities imply that $|F(u) - F(v)| \to 0$ as u,

$v \to \infty$, and that $G(s) < \epsilon/2$ when $s \geqq s_0$. Hence $\limsup |y(u) - y(v)| \leqq \epsilon$ and existence of $\lim y(s)$ follows.

If K is coercive and such that for each $x \,\varepsilon\, \mathfrak{K}_B$ the value of $\lim y(s)$ is independent of the value of $x(t)$ on each finite interval $0 \leqq t \leqq h$, then k is called *null.*

THEOREM 9.6. *Necessary and sufficient conditions that K be null are*

$$\int_0^\infty |K(s, t)| dt < \infty, \qquad s > 0, \tag{9.61}$$

$$\lim_{s\to\infty} \int_0^h K(s, t)x(t)dt = 0, \qquad h > 0,\ x \,\varepsilon\, \mathfrak{K}_B, \tag{9.62}$$

$$\lim_{h\to\infty} \limsup_{s\to\infty} \int_h^\infty |K(s, t)| dt = 0. \tag{9.63}$$

Moreover if K is null, then each $x \,\varepsilon\, \mathfrak{K}_B$ *is summable K to* 0.

Necessity follows from the fact that each null transformation is both coercive and multiplicative. Sufficiency is easily established by proving that each $x \,\varepsilon\, \mathfrak{K}_B$ is summable K to 0.

THEOREM 9.7. *Let C be a nonnegative real constant. In order that K may be such that the transform y(s) of each* $x \,\varepsilon\, \mathfrak{K}$ *for which* $\limsup |x(t)| < \infty$ *exists and has the property*

$$\limsup_{s\to\infty} |y(s)| \leqq C \limsup_{t\to\infty} |x(t)|, \tag{9.71}$$

it is necessary and sufficient that

$$\int_0^\infty |K(s, t)| dt < \infty, \qquad s > 0, \tag{9.72}$$

$$\lim_{s\to\infty} \int_0^h K(s, t)x(t)dt = 0, \qquad h > 0,\ x \,\varepsilon\, \mathfrak{K}, \tag{9.73}$$

and

$$\lim_{h\to\infty} \limsup_{s\to\infty} \int_h^\infty |K(s, t)| dt \leqq C. \tag{9.74}$$

For proof of necessity, the hypotheses imply that K is regular over the class of null functions $x \,\varepsilon\, \mathfrak{K}$ and hence that (9.72) and (9.73) hold. Necessity of (9.74) is implied by Theorem 8.2. Sufficiency is easily established. Theorem 9.7 remains true when $\mathfrak{K}$ is replaced by $\mathfrak{K}_B$.

Each theorem of this section has contained the condition

$$(9.81) \qquad \int_0^\infty |K(s, t)| dt < \infty, \qquad s > 0,$$

explicitly, and the condition

$$(9.82) \qquad \limsup_{s\to\infty} \int_0^\infty |K(s, t)| dt < \infty$$

either explicitly or implicitly. Each of these theorems remains true when (9.81) is deleted from the set of conditions and the definitions of regularity and so on are modified to correspond to the modified definition of summability under which $x(t)$ is called summable to L if

$$(9.83) \qquad y(s) = \int_0^\infty K(s, t)x(t)dt$$

exists for all sufficiently great s [that is, for all $s \geqq s_0$ where s_0 may depend upon the particular function x in (9.83)] and $\lim_{s\to\infty} y(s) = L$.* Proof of necessity for the new theorems is identical with that for the old, the theorems of §§7 and 8 having been made sufficiently general to make this true. Condition (9.82) implies existence of H such that

$$(9.84) \qquad \int_0^\infty |K(s, t)| dt < \infty, \qquad s > H,$$

and this plays the role of (9.81) in proof of sufficiency for the new theorems.

10. **Kernels which are step functions of t for each s; sequence to function transformations; matrix transformations.** It was pointed out in the introduction that a matrix transformation A can be identified with a kernel transformation K whose domain and range are confined to the class of (or a subclass of) step functions which are constant over each unit interval $n \leqq t < n+1$, $n = 0, 1, 2, \cdots$. This fact is, of course, significant, but it does not imply that from each theorem involving matrix transformations follows ipso facto a corresponding theorem involving kernel transformations whose domain and range are not restricted to classes of step functions. For example, matrix transformations include the identity transformation

$$y_s = \sum_{t=0}^\infty \delta_{s,t} x_t, \qquad s = 0, 1, 2, \cdots,$$

* This modified definition of "summable" is used in a study of regular transformations by J. D. Tamarkin, *On the notion of regularity of methods of summation of infinite series*, this Bulletin, vol. 41 (1935), pp. 241–243.

in which $\delta_{s,t}$ is 0 or 1 according as $t \neq s$ or $t=s$; but there is no kernel transformation

$$y(s) = \int_0^\infty K(s, t)x(t)dt$$

such that $y(s)=x(s)$, $0<s<\infty$, for all functions $x(s)$. It is likewise true that theorems involving kernel transformations do not ipso facto imply corresponding theorems involving matrix transformations.

The theorems of §9 involving kernel transformations are not merely analogous to, but actually imply, corresponding results for sequence-to-function transformations and for sequence-to-sequence matrix transformations.

A sequence $A_0(s)$, $A_1(s)$, $\cdots$ of complex-valued functions defined for $s>0$ determines the *sequence-to-function transformation*

$$(10.01) \qquad y(s) = \sum_{t=0}^{\infty} A_t(s)x_t, \qquad s > 0,$$

which associates with each sequence x_0, x_1, $\cdots$ for which the series converges for $s>0$ a transform $y(s)$. The transformation (10.01) is *regular* if $\lim x_t = L$ implies $\lim y(s) = L$, and other definitions are analogous to those for matrix transformations. Let x_t be any sequence of complex numbers. Then, assuming that $\int$ has the properties of §2,

$$(10.02) \qquad \sum_{t=0}^{n} A_t(s)x_t = \sum_{k=0}^{n-1} \int_k^{k+1} A_k(s)x_k dt, \qquad n = 0, 1, 2, \cdots.$$

If we define $K(s, t)$ and $x(t)$ by the formulas (in which k takes values $0, 1, 2, \cdots$)

$$(10.03) \qquad K(s, t) = A_k(s), \qquad k \leqq t < k+1, \; s > 0,$$

$$(10.04) \qquad x(t) = x_k, \qquad k \leqq t < k+1,$$

then K is a kernel according to the definition of §3, and $x \in \mathcal{K}$. The right member of (10.02) can then be written in the forms

$$(10.05) \qquad \sum_{k=0}^{n-1} \int_k^{k+1} K(s, t)x(t)dt = \int_0^n K(s, t)x(t)dt.$$

Setting, for all integers $n \geqq 0$, all real $h \geqq 0$, and all $s>0$,

$$F(n, s) = \sum_{t=0}^{n} A_t(s)x_t, \qquad G(h, s) = \int_0^h K(s, t)x(t)dt,$$

we see that, for each s, $G(h, s)$ is a linear function of h over each closed interval $n \leqq h \leqq n+1$. Moreover, since $F(n, s) = G(n, s)$, it follows that existence of either one of

$$(10.06) \qquad y^{(A)}(s) = \sum_{t=0}^{\infty} A_t(s) x_t = \lim_{n\to\infty} \sum_{t=0}^{n} A_t(s) x_t$$

or

$$(10.07) \qquad y^{(K)}(s) = \int_0^{\infty} K(s, t) x(t) dt = \lim_{h\to\infty} \int_0^{h} K(s, t) x(t) dt$$

implies existence of the other and the equality $y^{(A)}(s) = y^{(K)}(s)$.

Leaving consideration of other properties to the reader, we discuss regularity. It is clear from the above discussion of (10.06) and (10.07) that, if K is regular, and hence also regular over the class $\mathfrak{K}'$ of step functions $x(t)$ constant over each interval $n \leqq t < n+1$, then (10.07) is regular; and it is also clear that if (10.07) is regular, then K is regular over $\mathfrak{K}'$. Hence the first of the two next theorems is implied by the second.

THEOREM 10.1. *The sequence-to-function transformation*

$$(10.11) \qquad y(s) = \sum_{t=0}^{\infty} A_t(s) x_t, \qquad s > 0,$$

is regular if and only if the step kernel defined by (10.03) *is regular.*

THEOREM 10.2. *If K is a step kernel, $K(s, t)$ being independent of t over each interval $n \leqq t < n+1$, and if K is regular over the class $\mathfrak{K}'$ of step functions constant over each interval $n \leqq t < n+1$, then K is regular.*

To show that K is regular, let $x_0 \,\varepsilon\, \mathfrak{K}$ and $\lim x_0(t) = L$. We are to show that

$$(10.21) \qquad y_0(s) = \int_0^{\infty} K(s, t) x_0(t) dt, \qquad s > 0,$$

exists and $\lim y_0(s) = L$. Our hypothesis implies, by Theorem 7.15, that

$$(10.22) \qquad \int_0^{\infty} | K(s, t) | dt < \infty, \qquad s > 0,$$

and existence of (10.21) follows. Let $A_n(s)$ denote the value of $K(s, t)$ in the interval $n \leqq t < n+1$. The hypothesis $x_0 \,\varepsilon\, \mathfrak{K}$ implies that $x(t)$ is integrable over each interval $n \leqq t \leqq n+1$ corresponding to an integer n such that $A_n(s) \neq 0$ for some $s > 0$; the condition $x_0 \,\varepsilon\, \mathfrak{K}$ implies no condition whatever on $x_0(t)$ in an interval in which $A_n(s)=0$ for all $s>0$, since, in this case,

$$(10.23)\qquad \int_n^{n+1} K(s, t)x_0(t)dt = \int_n^{n+1} A_n(s)x_0(t)dt = 0,$$

however $x_0(t)$ is defined. Let $x_1(t)=x_0(t)$ in each interval $n\leqq t<n+1$ over which $x_0(t)$ is integrable, and let $x_1(t)=L$ for all other $t\geqq 0$. Then $x_1 \varepsilon \mathfrak{K}$ and $\lim x_1(t)=L$. Let

$$(10.24)\qquad x_2(t) = \int_n^{n+1} x_1(u)du, \qquad n \leqq t < n+1, n = 0, 1, \cdots.$$

Then $x_2 \varepsilon \mathfrak{K}'$ and $x_2\rightarrow L$. Starting with (10.21), we can show that

$$(10.25)\qquad \begin{aligned} y_0(s) &= \lim_{m\to\infty} \sum_{n=0}^{m-1} \int_n^{n+1} K(s, t)x_0(t)dt = \lim_{m\to\infty} \sum_{n=0}^{m-1} \int_n^{n+1} A_n(s)x_0(t)dt \\ &= \lim_{m\to\infty} \sum_{n=0}^{m-1} \int_n^{n+1} A_n(s)x_1(t)dt = \lim_{m\to\infty} \sum_{n=0}^{m-1} A_n(s)x_2(n) \\ &= \lim_{m\to\infty} \sum_{n=0}^{m-1} \int_n^{n+1} A_n(s)x_2(t)dt = \int_0^{\infty} K(s, t)x_2(t)dt. \end{aligned}$$

But since $x_2 \varepsilon \mathfrak{K}'$ and K is regular over $\mathfrak{K}'$, it follows that the last member of (10.25), and hence $y_0(s)$, converges to L as $s\rightarrow\infty$. This proves Theorem 10.2 and hence also Theorem 10.1.

THEOREM 10.3. *Necessary and sufficient conditions that the sequence-to-function transformation*

$$(10.31)\qquad y(s) = \sum_{t=0}^{\infty} A_t(s)x_t$$

be regular are

$$(10.32)\qquad \sum_{t=0}^{\infty} | A_t(s) | < \infty, \qquad s > 0,$$

$$(10.33)\qquad \limsup_{s\to\infty} \sum_{t=0}^{\infty} | A_t(s) | = M < \infty, \qquad s > 0,$$

$$(10.34)\qquad \lim_{s\to\infty} A_t(s) = 0, \qquad s = 0, 1, 2, \cdots,$$

$$(10.35)\qquad \lim_{s\to\infty} \sum_{t=0}^{\infty} A_t(s) = 1.$$

When K is the step kernel given by (10.03), the four criteria (5.11), (5.12), (5.13), and (5.14) for regularity of K are easily shown to be equivalent respectively to (10.32), (10.33), (10.34), and (10.35). Hence Theorem 10.3 is implied by Theorems 5.1 and 10.1.

It is often stated that the set of conditions consisting of

$$(10.36) \qquad \sum_{t=0}^{\infty} |A_t(s)| < M < \infty, \qquad s > 0,$$

(10.34) and (10.35) is necessary and sufficient for regularity of (10.31); this statement is false since the transformation

$$\begin{aligned} y(s) &= x_0, & s &= 0, \\ &= (1/s)x_0 + x_{[s]}, & s &> 0, \end{aligned}$$

is regular and fails to satisfy (10.36). On one hand, (10.36), (10.34), and (10.35) are sufficient for regularity, since (10.36) implies (10.32) and (10.33). On the other hand, the error in assuming that they are also necessary for regularity does not usually lead to false results on account of the fact that if

$$y(s) = \sum_{t=0}^{\infty} A_t(s)x_t, \qquad s > 0,$$

is a regular transformation which fails to satisfy (10.36), then there is a constant $H>0$ such that the transformation

$$y(s) = \sum_{t=0}^{\infty} A_t(s)x_t, \qquad s > H,$$

satisfies the condition

$$\sum_{t=0}^{\infty} |A_t(s)| < M < \infty, \qquad s > H,$$

which is analogous to (10.36).

If $A_{s,t}$ is a matrix of complex constants, and we define functions $A_t(s)$ by the formulas

$$A_t(s) = A_{n,t}, \qquad n \leqq s < n+1;\ n = 0, 1, \cdots,$$

it becomes apparent that the matrix transformation

$$(10.37) \qquad y_s = \sum_{t=0}^{\infty} A_{s,t}x_t$$

is regular if and only if (10.31) is regular. Hence, using Theorem 10.3, we see that necessary and sufficient conditions for regularity of matrix transformations are those stated at the beginning of §5.

Conditions that (10.31) and (10.37) be regular over other classes of sequences (for example, null sequences) are obtained by slight alteration of a few parts of this section.

11. **Scope of regular transformations.** Steinhaus* proved that no regular matrix A exists which evaluates all bounded sequences x_t. The following theorem extends this result to kernel transformations.

THEOREM 11.1. *If K is a regular transformation, then there is a function $x \in \mathfrak{K}$ for which $|x(t)| \leqq 1$ while the transform $y(s)$ has the property*

$$\limsup_{u,v\to\infty} |y(u) - y(v)| \geqq 2. \tag{11.11}$$

This theorem is an immediate result of Theorems 5.2 and 8.2.

THEOREM 11.2. *If, under a given definition of integral having the properties listed in §2, $x(t)$ is a bounded function which is integrable over each finite interval $0 \leqq t \leqq h$, then there is a regular transformation K involving the given definition of integral such that $x(t)$ is summable K.*

The manner in which we construct a regular transformation which evaluates $x(t)$ indicates that many such transformations can be constructed. The hypotheses of Theorem 10.2 imply that the sequence $z(0), z(1), \cdots$ of complex numbers defined by

$$z(n) = \int_n^{n+1} x(t)dt, \qquad n = 0, 1, 2, \cdots,$$

is bounded. Hence there is a sequence $0 = n_1 < n_2 < \cdots$ of indices such that $\lim_{p\to\infty} z(n_p)$ exists. The transformation

$$y(s) = \int_{n_p}^{n_{p+1}} x(t)dt, \qquad n_p \leqq s < n_{p+1},$$

is obviously regular and evaluates $x(t)$ to $\lim z(n_p)$.

Theorems 11.1 and 11.2 indicate that the fitting of regular transformations to functions is like the fitting of shoes to men. No one pair of shoes will fit all men; but any man can obtain from a shoemaker many different styles of shoes which will fit him.

12. **Conclusion.** Let E and $\mathcal{E}$ be point sets, and let integration be so defined that for at least some complex-valued functions $f(t)$ defined

* H. Steinhaus, *Some remarks on the generalizations of the notion of limit* (in Polish), Prace Matematyczno-Fizyczne, vol. 22 (1921), pp. 121–134.

for $t \varepsilon E$, the symbol

$$\int_E f(t)dt \tag{12.1}$$

represents a complex number corresponding to f. There is now the possibility that if the integral has appropriate properties, and if $K(s, t)$ is an appropriate complex-valued function defined for $s \varepsilon \mathcal{E}$, $t \varepsilon E$, then the transformation

$$y(s) = \int_E K(s, t)x(t)dt, \qquad s \varepsilon \mathcal{E}, \tag{12.2}$$

is significant. In case the set $\mathcal{E}$ is one (not necessarily in any euclidean space or even in a metric space) in which the notion of limit point is defined, and s_0 is a limit point of $\mathcal{E}$, then $x(t)$ may be called *summable* (12.2) to L if $y(s)$ exists for $s \varepsilon \mathcal{E}$ and $y(s) \to L$ as $s \to s_0$ over $\mathcal{E}$. In case t_0 is a limit point of E, the number $\lim_{s \to s_0} y(s)$ may (when it exists) be regarded as a *generalized limit* of $x(t)$ as $t \to t_0$ over E. The transformation (12.2) is *regular over a class C* of functions if $x(t)$ is summable to L whenever $x \varepsilon C$ and $x(t) \to L$ as $t \to t_0$ over E. Other properties which (12.2) may have or fail to have can be similarly defined.

The transformation (12.2) and the associated method of summability determined by t_0 and s_0 reduce to the form we have studied when $\mathcal{E}$ is the set $0 < s < \infty$, E is the set $0 < t < \infty$, and t_0 and s_0 are the symbolic limit points $+\infty$.

It is apparent that the role of t in the theory of the transformation (12.2) lies far deeper than the role of s. In so far as the theory of K which we have covered in this address is concerned, the difference between K and the transformation

$$y(s) = \int_0^\infty K(s, t)x(t)dt, \qquad s \varepsilon \mathcal{E}, \tag{12.3}$$

with $s \to s_0$ instead of $s \to \infty$, is trivial. For example, (12.3) may furnish a regular generalization of $\lim_{t\to\infty} x(t)$ when $s \to s_0$ over a set $\mathcal{E}$ which is a bounded interval in euclidean space of one dimension, a "curve" in a plane, or a non-measurable set in 3-space.

On the other hand the transformation

$$y(s) = \int_E K(s, t)x(t)dt, \qquad s > 0, \tag{12.4}$$

in which $s_0 = +\infty$, assumes entirely different forms according to the character of E and t_0 and the definition of integral used. In case E is the interval $a<t<\infty$, where a is a constant not 0, the transformation (12.4) differs only in an obvious and trivial way from K. Easy modifications of our theory of K cover some cases in which E is a curve other than the interval $a<t<\infty$ in a plane. These examples furnish generalizations of lim $x(t)$ as $t\to t_0$ over a curve.

For an example of a different character, let E denote the set of points of a plane which are interior to the unit circle with center at the origin, and let t_0 be a point on the unit circle. For each $h>0$, let E_h be the set of points interior to the circle with center at t_0 and radius h. For each $h>0$, let

$$\int_{E-E_h} f(t)dt$$

denote the Lebesgue double integral of f over $(E-E_h)$ when the integral exists, and let

$$\int_E f(t)dt = \lim_{h\to 0} \int_{E-E_h} f(t)dt$$

when the limit exists. The transformation (12.4) then furnishes a generalized definition of the limit of $x(t)$ as $t\to t_0$ over the interior of the unit circle.

It may be possible to give a theory of (12.2) covering at the same time all of the special cases which we have mentioned and many others. Such a theory would be more impressive and would have wider application than the theory we have given. However the details necessary to make the more general theory precise would have made our discussion more cumbersome, and might have obscured the real purpose of this address.

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