# REGULARITY OF FUNCTION-TO-FUNCTION TRANSFORMATIONS*

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1. **Introduction.** In a recent note Hill† considered the transformation

$$U_x(f) = \int_0^\eta K(x, y)f(y)dy \tag{1}$$

operating on the class $\mathfrak{B}_1$ of measurable, essentially bounded, complex-valued functions $f$ of one real variable $y$ defined for $0<y<\eta$, and satisfying the condition that $\lim_{y\to\eta} f(y)$ exists. The kernel $K(x, y)$ is defined for $0<x<\xi$, $0<y<\eta$, and the integral is interpreted in the Lebesgue sense. Hill derived a set of conditions on $K(x, y)$ necessary and sufficient that the transformation (1) be *regular* on $\mathfrak{B}_1$, that is, that for every $f$ in $\mathfrak{B}_1$, $U_x(f)$ be defined for all $x$, and $\lim_{x\to\xi} U_x(f)=\lim_{y\to\eta} f(y)$.

In §2 of the present paper we generalize Hill's results for a transformation on a class $\mathfrak{B}_m$ of bounded measurable functions of $m$ real variables to a class of functions of $n$ real variables. This transformation can be expressed in the form (1) with $x$ standing for $x^1, x^2, \cdots, x^n$ and $y$ for $y^1, y^2, \cdots, y^m$. In §3 we define for each kernel $K(x, y)$ its domain of regularity $\mathfrak{R}$ as the largest class of functions on which (1) is regular, and we determine some conditions necessary and sufficient that a function $f$ be in $\mathfrak{R}$. We employ these results in §4 to derive conditions on $K(x, y)$ necessary and sufficient for the transformation (1) to be regular on certain classes of functions more inclusive than $\mathfrak{B}_m$. Finally, in §5, we consider several particular cases of the problem of determining a kernel with a specified class of functions as its domain of regularity.

2. **Hill's theorem in many variables.** We use the following notation throughout this paper: $x$ stands for the ordered set of $n$ real variables $x^1, x^2, \cdots, x^n$, and $y$ for $y^1, y^2, \cdots, y^m$; for $0, 0, \cdots, 0$ we write $0$. The equality $a=b$ means that for each $j$, $a^j=b^j$; $a<b$ that for each $j$, $a^j<b^j$; $a\not< b$ that for at least one $j$, $a^j\geqq b^j$; and $a>b$ that $b<a$. We define the *interval* $(a, b)$ as the set of points $c$ such that $a<c<b$.

* Presented to the Society, September 6, 1938.

† J. D. Hill, *A theorem in the theory of summability*, this Bulletin, vol. 42 (1936), pp. 225–228.

We consider only finite-valued functions defined on a given interval $0<y<\eta$ or $0<x<\xi$ where we allow $\xi$ and $\eta$ to be chosen so that any of the $\xi^i$ or $\eta^i$ may be either positive real numbers or may be infinite; in either case if $y<\eta$, then every $y^i$ is finite.

We let $\lim_{y\to\eta} f(y)$ stand for the multiple limit, if it exists,* of $f(y)=f(y^1, \cdots, y^m)$ as the $y^i$ tend to the $\eta^i$ simultaneously but independently. We interpret all integrals as $m$-dimensional Lebesgue integrals which, by Fubini's theorem,† can be evaluated, whenever they exist, by iterated integration.

DEFINITION 1. *On a given interval* $(0, \eta)$ *let* $\mathfrak{W}_m$ *be the class of complex-valued functions,* $f(y)=f_1(y)+if_2(y)$, *satisfying the conditions that* (a) $f_1$ *and* $f_2$ *are measurable on* $(0, \eta)$, *and* (b) $f(y)$ *tends to a finite limit* $L_f$ *as* $y$ *tends to* $\eta$.

*Let* $\mathfrak{B}_m$ *be the subclass of* $\mathfrak{W}_m$ *such that* $f \in \mathfrak{B}_m$ *if and only if* $f$ *is essentially bounded on* $(0, \eta)$.

DEFINITION 2. *For a given* $\xi$ *and* $\eta$ *let* $K(x, y)=K_1(x, y)+iK_2(x, y)$ *be a complex-valued function defined for* $0<x<\xi$ *and* $0<y<\eta$ *and such that for each* $x$, $K(x, y)$ *is a measurable function of* $y$. *Let*

$$(1)\qquad \begin{aligned} U_x(f) &= \int_0^\eta K(x, y)f(y)dy \\ &= \int_0^\eta [K_1f_1 - K_2f_2] + i\int_0^\eta [K_1f_2 + K_2f_1]. \end{aligned}$$

*Let* $\mathfrak{W}_m'$ *be any subclass of* $\mathfrak{W}_m$; $K(x, y)$ *is said to define a regular transformation on* $\mathfrak{W}_m'$, *or is said to be regular on* $\mathfrak{W}_m'$, *if and only if for each* $f$ *in* $\mathfrak{W}_m'$, (a) *for each* $x$ *in* $(0, \xi)$, $U_x(f)$ *exists, and* (b) $\lim_{x\to\xi} U_x(f)=L_f$.

THEOREM 1. *For* $K(x, y)$ *to be regular on* $\mathfrak{B}_m$, *it is necessary and sufficient that* (a) $\int_0^\eta K(x, y)dy \to 1$ *as* $x\to\xi$, (b) *for every measurable set* $H\subset(0, \eta)$ *not having* $\eta$ *as a limit point* $\int_H K(x, y)dy\to 0$ *as* $x\to\xi$, (c) *for each* $x$ *in* $(0, \xi)$, $\int_0^\eta |K(x, y)|dy$ *exists, and* (d) *there exist a number* $M$ *and a point* $X<\xi$ *such that* $\int_0^\eta |K(x, y)|dy<M$ *for all* $x$ *in* $(X, \xi)$.

The proof of this theorem will not be given here since the notation which makes it possible to state the theorem in this form also suggests the necessary minor modifications in Hill's proof of the special case in which $n=m=1$.

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* "Exists" shall mean "is a finite real or complex number."

† See S. Saks, *Theory of the Integral*, p. 77.

3. **The domain of regularity of a transformation.** We state the following definitions:

DEFINITION 3. *For a given kernel* $K(x, y)$ *the domain of regularity* $\mathfrak{R}$ *of* (1) *is the class of all functions* $f$ *which satisfy the conditions*:

(2) $f \in \mathfrak{W}_m$.

(3) *For every* $x$ *in* $(0, \xi)$, $U_x(f)$ *exists.*

(4) $\lim_{x\to\xi} U_x(f) = L_f$.

DEFINITION 4. *For any function* $f(y)$ *and any real number* $b$ *let* $B = E_y[|f(y)| > b]$, *and let* $f_b(y) = 0$ *if* $y \in B$, $f_b(y) = f(y)$ *otherwise.*

We remark that if $f(y)$ is in $\mathfrak{W}_m$, and if $b > |L_f|$, then $f_b(y)$ is in $\mathfrak{B}_m$ and $L_f = L_{f_b}$.

THEOREM 2. *If* $K(x, y)$ *is regular on* $\mathfrak{B}_m$, *a function* $f$ *belongs to* $\mathfrak{R}$ *if and only if it satisfies the conditions* (2) *and* (3), *and the following condition*:

(5) *For every* $b > |L_f|$,

$$\int_B K(x, y)f(y)dy \to 0 \quad as \quad x \to \xi.$$

PROOF. We need only show (4) and (5) equivalent here. We have for any $b > |L_f|$,

$$U_x(f) = U_x(f_b) + \int_B K(x, y)f(y)dy.$$

But $f_b \in \mathfrak{B}_m$ and $L_{f_b} = L_f$; so $\lim_{x\to\xi} U_x(f_b) = L_f$, and in order that $U_x(f) \to L_f$ as $x \to \xi$ it is necessary and sufficient that $\int_B K(x, y)f(y)dy \to 0$ as $x \to \xi$.

THEOREM 3. *If* $K(x, y)$ *is regular on* $\mathfrak{B}_m$, *a function* $f$ *belongs to* $\mathfrak{R}$ *if and only if it satisfies the conditions* (2) *and* (3), *and the additional condition*:

(6) *For every* $\epsilon > 0$, *there exist a number* $b > |L_f|$ *and a point* $X < \xi$ *such that for all* $x$ *in* $(X, \xi)$

$$\left| \int_B K(x, y)f(y)dy \right| < \epsilon.$$

PROOF. Here we need only show (4) and (6) equivalent. If $K$ is regular on $\mathfrak{B}_m$ and $f$ satisfies (4), by the preceding theorem, $f$ satisfies (5) and so obviously (6).

For any $b > |L_f|$,

$$| U_x(f) - L_f | \leqq \left| \int_B Kf \right| + | U_x(f_b) - L_f |.$$

If $f$ satisfies (6) and is regular on $\mathfrak{B}_m$, then for any $\epsilon > 0$, there exist a $b > |L_f|$ and an $X < \xi$ such that the right-hand side is less than $\epsilon$ for $x$ in $(X, \xi)$. But the left-hand side does not depend on $b$; so $U_x(f) \to L_f$ as $x \to \xi$ and (6) implies (4).

4. **Conditions for regularity on other subclasses of $\mathfrak{W}_m$.** In this section we consider conditions on $K(x, y)$ necessary and sufficient that the transformation (1) be regular on certain classes containing $\mathfrak{B}_m$.

DEFINITION 5. *Let $\mathfrak{S}_m$ be the subclass of $\mathfrak{W}_m$ such that $f \in \mathfrak{S}_m$ if and only if for every $Y < \eta$, $f(y)$ is L-integrable over* $(0, Y)$. *Let $\mathfrak{T}_m$ be the subclass of $\mathfrak{S}_m$ such that $f \in \mathfrak{T}_m$ if and only if there exists a point $Y < \eta$ such that $f$ is essentially bounded in the region* $(0, \eta) - (0, Y)$. *Let $\mathfrak{R}_m$ be the subclass of $\mathfrak{W}_m$ such that $f \in \mathfrak{R}_m$ if and only if for every $Y < \eta$, $f(y)$ is essentially bounded on* $(0, Y)$.

We have the obvious relations that $\mathfrak{W}_m \supset \mathfrak{S}_m \supset \mathfrak{T}_m \supset \mathfrak{B}_m$, that $\mathfrak{S}_m \supset \mathfrak{R}_m \supset \mathfrak{B}_m$, that $\mathfrak{R}_1 = \mathfrak{B}_1$, and that $\mathfrak{T}_1 = \mathfrak{S}_1$.

THEOREM 4. *For $K(x, y)$ to be regular on $\mathfrak{T}_m$ it is necessary and sufficient that $K(x, y)$ be regular on $\mathfrak{B}_m$, and satisfy the following condition:*

(7) *For every $Y < \eta$,* (a) *for each $x$,*

$$F_Y(x) = \operatorname*{ess\,sup}_{0<y<Y} | K(x, y) |$$

*exists, and* (b) *there exists a number $M$ and a point $X < \xi$ such that $F_Y(x) < M$ for every $x$ in* $(X, \xi)$.

PROOF OF NECESSITY. The first condition is necessary as $\mathfrak{B}_m$ is contained in $\mathfrak{T}_m$. If (7a) is not satisfied, there exist $Y < \eta$ and $x < \xi$ such that $K(x, y)$ is not essentially bounded on $(0, Y)$. Hence there exists a function $f(y)$, L-integrable in $(0, Y)$ and zero in $(0, \eta) - (0, Y)$, such that

$$U_x(f) = \int_0^Y K(x, y) f(y) dy$$

does not exist. But this function is in $\mathfrak{T}_m$; so (7a) is necessary.

For the necessity of (7b) we again modify Hill's proof. Fix any $Y < \eta$, and let $\mathfrak{L}_m$ be the subclass of $\mathfrak{T}_m$ such that $f$ is in $\mathfrak{L}_m$ if and only if $f(y) = 0$ for $y$ in $(0, \eta) - (0, Y)$. Then with the usual definition of equality, addition, and multiplication by real numbers, $\mathfrak{L}_m$ is a Banach space with norm given by $\|f\| = \int_0^Y |f(y)| dy$. For each fixed $x$,

$$U_x(f) = \int_0^Y K(x, y) f(y) dy$$

is a linear functional on $\mathfrak{L}_m$ with norm $\|U_x\| = F_Y(x)$.

For any sequence $\{x_k\}$ tending to $\xi$, let $U_{x_k} = U_k$. Then for every $f$ in $\mathfrak{L}_m$, $U_k(f) \to 0$ as $k \to \infty$ since $K(x, y)$ is regular on $\mathfrak{T}_m$. Hence, for every $f$ in $\mathfrak{L}_m$, the sequence $\{|U_k(f)|\}$ is bounded; so, by a theorem of Banach,* the sequence of norms $\{\|U_k\|\}$ is bounded. Since $\{x_k\}$ is any sequence tending to $\xi$, and since $\|U_x\| = F_Y(x)$, there exist $M$ and $X < \xi$ such that $F_Y(x) < M$ for $x$ in $(X, \xi)$.

Proof of sufficiency. For any $f$ in $\mathfrak{T}_m$, there exist $M$ and $Y < \eta$ such that $|f(y)| < M$ for $y$ in $(0, \eta) - (0, Y)$. Then for any $b > M$ the set $B$ lies in $(0, Y)$ and

$$\int_0^\eta |K(x, y) f(y)| dy \leqq \int_0^\eta |K f_b| + \int_B |Kf|$$
$$\leqq b \int_0^\eta |K| + F_Y(x) \int_B |f|.$$

Both integrals in the last term exist; so $\int_0^\eta |Kf|$ exists and $U_x(f)$ therefore exists for each $x$. Now $|\int_B Kf| \leqq F_Y(x) \int_B |f|$, and this can be made arbitrarily small by taking $x$ in $(X, \xi)$ and $b$ sufficiently large. By Theorem 3, $f$ is in $\mathfrak{K}$; therefore we have $\mathfrak{T}_m \subset \mathfrak{K}$, and $K(x, y)$ is regular on $\mathfrak{T}_m$.

Definition 6. *In* $(0, \eta)$ *a set* $E$ *is said to be essentially contained in a set* $F$ *if and only if* $m(E - EF) = 0$.

Definition 7. *For any given kernel* $K(x, y)$ *for each* $x < \xi$ *and each* $Y < \eta$ *let* $Y_x$ *be the point such that* (a) *the sum of the intervals* $(0, Y_x)$ *and* $(Y, \eta)$ *essentially contains* $E_y[K(x, y) \neq 0]$, *and* (b) *if* $(0, Y_0) + (Y, \eta)$ *essentially contains* $E_y[K(x, y) \neq 0]$, *then* $(0, Y_x) \subset (0, Y_0)$.

Theorem 5. *For* $K(x, y)$ *to be regular on* $\mathfrak{S}_m$ *it is necessary and sufficient that* $K(x, y)$ *be regular on* $\mathfrak{T}_m$ *and satisfy the condition*:

(8) *For every* $Y < \eta$, (a) *for every* $x$ *in* $(0, \xi)$, $Y_x < \eta$, *and* (b) *there exist* $Y' < \eta$ *and* $X < \xi$ *such that* $Y_x < Y'$ *for all* $x$ *in* $(X, \xi)$.

Proof of necessity. If $m = 1$ condition (8) is satisfied by every kernel $K(x, y)$, since, for each $Y$ and $x$, there is always a $Y_x \leqq Y$ such

* S. Banach, *Théorie des Opérations Linéaires*, p. 80, Theorem 5.

that $(0, Y_x)+(Y, \eta)$ essentially contains $E_y[K(x, y)\neq 0]$. (8b) can then be satisfied by taking $X=0$ and $Y'\geqq Y$.

For $m>1$ fix any $Y<\eta$; if (8a) is not satisfied for some $x$, then there exist a sequence $\{Y_k\}\rightarrow\eta$ and a function $f_0$ in $\mathfrak{S}_m$ such that $\int_0^{Y_k}Kf_0>k$. Then $U_x(f_0)$ does not exist; hence (8a) is necessary.

If (8b) is not satisfied, there is a $Y<\eta$ such that for each $Y'<\eta$ and $X<\xi$ there is an $x$ in $(X, \xi)$ such that $Y_x \not< Y'$. Hence there exists a sequence $\{x_k\}\rightarrow\xi$ such that, if we let $Y_{x_k}=Y_k$, $(0, Y_k)-(0, Y_{k-1})$ is not empty. Moreover $\{x_k\}$ can be so chosen that for some $j\leqq m$, we have $Y_k^j\rightarrow\eta^j$ and $Y_k^j>Y_{k-1}^j$. Then there exists a function $f$ in $\mathfrak{S}_m$ such that $f(y)=0$ for $y$ in $(Y, \eta)$ and such that for each $k$,

$$U_k(f)=\int_0^{Y_k}K(x_k, y)f(y)dy>1.$$

This is possible because for each $k$ there is a part of $(0, Y_k)$ which is in none of the sets $(0, Y_p)+(Y, \eta)$ for any $p<k$, and in this part of $(0, Y_k)$ there is a set of positive measure on which $K(x_k, y)\neq 0$. Then $L_f=0$, but $U_k(f)>1$ for all $k$ so $U_x(f)$ cannot tend to 0 as $x\rightarrow\xi$. This contradiction proves (8b) necessary.

Proof of sufficiency. Condition (8a) and the condition that $K$ be regular on $\mathfrak{T}_m$ insure that $U_x(f)$ exists for all $f$ in $\mathfrak{S}_m$ and all $x$ in $(0, \xi)$. For any $f$ in $\mathfrak{S}_m$, there exists $Y<\eta$ and $M$ such that $|f(y)|<M$ for $y$ in $(Y, \eta)$, because $f(y)\rightarrow L_f$ as $y\rightarrow\eta$. Then by (8b) there exist $Y'<\eta$ and $X<\xi$ such that for each $x$ in $(X, \xi)$, $K(x, y)=0$ almost everywhere outside $(0, Y')+(Y, \eta)$. Let $f_0(y)=f(y)$ for $y$ in $(0, Y')+(Y, \eta)$ and $f_0(y)=0$ elsewhere. Then $f_0$ is in $\mathfrak{T}_m$ since it is in $\mathfrak{S}_m$ and $|f_0(y)|<M$ for $y$ in $(0, \eta)-(0, Y')$. Then $L_{f_0}=L_f$; so $U_x(f_0)\rightarrow L_f$ as $x\rightarrow\xi$. But for $x$ in $(X, \xi)$, $U_x(f)=U_x(f_0)$; so $\lim_{x\rightarrow\xi} U_x(f)=L_f$ and $K(x, y)$ is regular on $\mathfrak{S}_m$.

Theorem 6. *For $K(x, y)$ to be regular on $\mathfrak{R}_m$ it is necessary and sufficient that $K(x, y)$ be regular on $\mathfrak{B}_m$ and satisfy condition* (8).

The proof is almost the same as that of Theorem 5.

We remarked that the classes $\mathfrak{T}_1$ and $\mathfrak{S}_1$ are identical. It follows that (8) is not a restriction on $K(x, y)$ for $m=1$. From Theorems 4 and 5 we see that for $m>1$, the generalized arithmetic mean transformation given by $K(x, y)=1/x^1x^2\cdots x^m$ if $0<y^j<x^j$ for all $j\leqq m$, and $K(x, y)=0$ otherwise, is regular on $\mathfrak{T}_m$ but not on $\mathfrak{S}_m$ or $\mathfrak{R}_m$. This result resembles that of Kojima* on summability of multiple sequences.

* Kojima, *On the theory of double sequences*, Tôhoku Mathematical Journal, vol. 21 (1922), pp. 3–14.

5. **Kernels with specified domains of regularity.** In this section we consider the problem of defining a kernel having as domain of regularity one of the classes $\mathfrak{W}_m$, $\mathfrak{S}_m$, and $\mathfrak{V}_m$. For convenience we shall consider $n=m$ and all $\eta^i$ and $\xi^i$ infinite, but analogous results hold for any choice of these limits.

THEOREM 7. *For no $m$ is $\mathfrak{W}_m$ the domain of regularity of any kernel.*

PROOF. If for each $f$ in $\mathfrak{W}_m$ and each $x$ in $(0, \xi)$, $U_x(f)$ exists, then for each $x$, $K(x, y)=0$ for almost all $y$ in $(0, \eta)$. Hence $\int_0^\eta K(x, y)dy=0$, so

$$\lim_{x\to\xi}\int_0^\eta K(x, y)dy = 0;$$

but $\mathfrak{W}_m \supset \mathfrak{V}_m$, so by Theorem 1,

$$\lim_{x\to\xi}\int_0^\eta K(x, y)dy = 1.$$

THEOREM 8. *The kernel*

$$K(x, y) = \begin{cases} \prod_{j=1}^m (1-x^j) \text{ if, for every } j,\ 0<x^j<1 \text{ and } 0<y^j<1/(1-x^j), \\ 2^m\prod_{j=1}^m 1/x^j \text{ if, for some } j,\ 1\leqq x^j<\infty \text{ and for every } j, \\ \quad x^j/2<y^j<x^j, \\ 0 \text{ otherwise,} \end{cases}$$

*has $\mathfrak{S}_m$ for domain of regularity.*

PROOF. For each $x$, $\int_0^\eta K(x, y)dy=1$, and for all $x$ and $y$, $0\leqq K(x, y)\leqq 1$. From this it is easily verified that $K$ satisfies the conditions of Theorem 5 so that $\mathfrak{S}_m \subset \mathfrak{K}$. If $f$ is not in $\mathfrak{S}_m$, then there exists a $Y<\eta$ such that $\int_0^Y f$ does not exist; also there is an $x_0$ such that for each $j$, $0<x_0{}^j<1$ and $1/(1-x_0{}^j)\geqq Y^j$ so that

$$\int_0^Y K(x_0, y)f(y)dy$$

does not exist and $f$ is not in $\mathfrak{K}$. Hence $\mathfrak{K}=\mathfrak{S}_m$.

THEOREM 9. *There is a kernel having $\mathfrak{V}_m$ as its domain of regularity.*

PROOF. Let $\{G\}$ be the class of open sets in $(0, \eta)$; then there is a one-to-one correspondence between the class of points of the interval $0<s<1$ and the class of sequences $\{G_i\}$ of sets of $\{G\}$. Let $g_i(y)$ be the characteristic function of $G_i$. For each $s$ and corresponding sequence $\{G_i\}$ let $f_s(y)=\sum_{i=0}^\infty g_i(y)$; then define

$$K(x, y) = \begin{cases} f_s(y) \text{ if } x^1 = s < 1 \text{ and } \int_0^\eta f_s < \infty, \\ \prod_{j=1}^{m} 1/x^j \text{ if } x^1 \geqq 1 \text{ and } 0 < y^j < x^j \text{ for all } j, \\ 0 \text{ otherwise.} \end{cases}$$

Then $K$ is regular on $\mathfrak{V}_m$; so $\mathfrak{R} \supset \mathfrak{V}_m$. If $f \in \mathfrak{W}_m - \mathfrak{V}_m$, either there is a $b > 0$ such that $B$ is of finite measure and $\int_B f$ exists or not; if there is no such $b$, $|f|$ dominates a function with this property. Hence there is a function $f_0(y)$, zero if $y$ is not in $B$, such that $\int_B f_0$ exists and $\int_B f_0 f$ does not. Then there is an $s < 1$ such that $f_s$ dominates $f_0$ but $\int_0^\eta f_s < \infty$. Hence there is an $x$ such that $U_x(f)$ does not exist; so $\mathfrak{V}_m \supset \mathfrak{R}$.

For $m \geqq 2$ this construction can be modified to give a kernel with $\mathfrak{N}_m$ as domain of regularity.

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# BOUNDED SELF-ADJOINT OPERATORS AND THE PROBLEM OF MOMENTS*

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It is known that there exists a close connection between the theory of moments and Jacobi matrices on one side, and the theory of self-adjoint operators in Hilbert space on the other. This connection has been thoroughly investigated by Stone in the tenth chapter of his textbook on Hilbert space.† The solution of both the moment problem and the spectral resolution of self-adjoint operators relies on the possibility of representing a class of analytic functions with positive imaginary parts in the upper half-plane by *Stieltjes* integrals of the form

$$\int_{-\infty}^{+\infty} \frac{d\rho(\lambda)}{\lambda - z}.$$

However, the spectral resolution requires the representation of more general functions than those involved in the solution of the problem of moments. The *bounded* self-adjoint operators do not, and I wish to show that the spectral theorem for *bounded* operators can be deduced immediately from the well known theorems concerning the problem of moments. Let $H$ be a bounded self-adjoint operator, $f$ an element of the Hilbert space, $R_z = (H - zI)^{-1}$ the resolvent of $H$. It

* Presented to the Society, September 6, 1938.

† M. H. Stone, *Linear Transformations in Hilbert Space and their Applications to Analysis*, American Mathematical Society Colloquium Publications, vol. 15, New York, 1932. The notations of this textbook are used throughout the present paper.