EXPANSION OF FUNCTIONS IN SOLUTIONS OF FUNCTIONAL EQUATIONS*

LEONARD BRISTOW

1. **Introduction.** In analysis a number of functional equations have solutions of the form

$$x^r \sum_{s=0}^{\infty} \alpha_{s,r} x^s. \tag{1}$$

Examples are (a) linear differential equations with a regular singular point at the origin, (b) the Volterra homogeneous integral equation with a regular singularity, (c) the linear q-difference equation, (d) the Fuchsian equation of infinite order. There are many others including mixed q-difference and differential equations.

Consider the equation

$$L(x, \lambda) \rightarrow y = 0 \tag{2}$$

where λ is a parameter and $L(x, \lambda)$ is an operator with the following property:

$$L(x, \lambda) \rightarrow x^p = x^p f(x, p, \lambda) = x^p \sum_{\mu=0}^{\infty} f_\mu(p, \lambda) x^\mu, \tag{3}$$

the series converging for $|x| \leqq N < r$ for all values of p, which may be a complex number. The purpose of this paper is to consider under what conditions a set of values $\{\lambda_m\}$, $(m = 0, 1, 2, \cdots)$, can be determined so that for $\lambda = \lambda_m$ there will exist a solution of the form

$$\begin{aligned} y_{m+\sigma}(x) &= x^{m+\sigma} \sum_{s=0}^{\infty} \alpha_s^{(m+\sigma)} x^s = \sum_{s=0}^{\infty} \alpha_s^{(m+\sigma)} x^{m+\sigma+s} \\ &= x^{m+\sigma} \{ \alpha_0^{(m+\sigma)} h_m(x) \} \end{aligned} \tag{4}$$

such that an arbitrary function $x^\sigma f(x)$, $f(x)$ being analytic for $|x| < \rho$, can be expanded in a series

$$x^\sigma f(x) = \sum_{m=0}^{\infty} a_m y_{m+\sigma}(x) \tag{5}$$

which converges and represents the function in some region. For the

* Presented to the Society, October 29, 1938.

sake of simplicity we will consider $\sigma=0$; the extension to σ any number will be evident. In order to do this we will make use of the following theorem due to Mrs. Gertrude Stith Ketchum* which will be denoted by Theorem K:

THEOREM K. *Consider the set of functions* $g_m(x)=x^m\{1+h_m(x)\}$ *where* $h_m(0)=0$ *and* $h_m(x)$ *is analytic for* $|x|<r$. *If there exists an* $M_{N,m}$ *such that* $|h_m(x)|\leqq M_{N,m}$ *for a given positive* $N<r$ *and* $|x|\leqq N$, *and if* $\limsup M_{N,m}=K_N$ *(finite), then any function analytic for* $|x|<\rho$ *has a unique uniformly convergent expansion* $f(x)=\sum_{m=0}^{n=\infty}\alpha_m g_m(x)$ *for* $|x|\leqq R<G$, *where*

$$G=\min\left\{p, \max_N N[1+K_N]^{-1}\right\},$$

and the expansion converges absolutely for $|x|<G$.

We seek then to determine conditions under which the $y_m(x)$, $(m=0, 1, \cdots)$, will satisfy the conditions of this theorem.

2. **Sufficient conditions.** Operating formally upon both sides of (4) $(\sigma=0)$ with the operator $L(x, \lambda)$ we get

$$L(x,\lambda)\rightarrow y_m(x)=\sum_{s=0}^{\infty}\alpha_s^{(m)}x^{m+s}\sum_{\mu=0}^{\infty}f_\mu(m+s,\lambda)x^\mu=0. \tag{6}$$

Equating the coefficients of powers of x to zero we get the following set of equations for the determination of the $\alpha_s^{(m)}$, $(s=0, 1, \cdots)$:

$$\begin{aligned}
&\alpha_0^{(m)}f_0(m,\lambda)=0,\\
&\alpha_1^{(m)}f_0(m+1,\lambda)+\alpha_0^{(m)}f_1(m,\lambda)=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\alpha_{s+1}^{(m)}f_0(m+s+1,\lambda)+\alpha_s^{(m)}f_1(m+s,\lambda)+\cdots+\alpha_0^{(m)}f_{s+1}(m,\lambda)=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}\tag{7}$$

If $\alpha_0^{(m)}\neq 0$, then $f_0(m,\lambda)=0$ to give a solution of the desired form. Suppose that

$$f_0(m,\lambda_m)=0, \qquad m=0,1,\cdots, \tag{8}$$

determines a set of characteristic values $\{\lambda_m\}$, and further suppose that

$$f_0(p,\lambda_m)\neq 0, \qquad m\neq p. \tag{9}$$

* Transactions of this Society, vol. 40 (1935), pp. 208–224.

The coefficients $\alpha_{s+1}^{(m)}$ can be determined for $s=0, 1, \cdots$. Since $\alpha_0^{(m)}$ is arbitrary, we will choose it to be unity. By the method of Frobenius* we get the following set of inequalities:

$$\text{(10)} \qquad \left| \alpha_{s+1}^{(m)} \right| \leqq A_{s+1}^{(m)} \leqq A_s \left\{ \frac{M_N(m+s, \lambda_m) + \left| f_0(m+s, \lambda_m) \right|}{\left| f_0(m+s+1, \lambda_m) \right|} \right\}$$

$$= A_s^{(m)} P(m, s),$$

where

$$\text{(11)} \qquad \begin{aligned} A_{s+1}^{(m)} = \Big\{ \left| \alpha_s^{(m)} \right| M_N(m+s, \lambda_m) + \left| \alpha_{s-1}^{(m)} \right| M_N(m+s-1) N^{-1} + \cdots \\ + \left| \alpha_0^{(m)} \right| M_N(s) N^{-s} \Big\} \left| f_0(m+s+1, \lambda_m) \right|^{-1} \end{aligned}$$

and $M_N(m+s, \lambda_m)$ are such that

$$\text{(12)} \qquad \left| \frac{d}{dx} f(x, m+s, \lambda_m) \right| \leqq M_N(m+s, \lambda_m).$$

It is evident that

$$\text{(13)} \qquad \left| h_m(x) \right| \leqq F_m(x),$$

where

$$\text{(14)} \qquad F_m(x) = \sum_{s=1}^{\infty} A_s^{(m)} \left| x \right|^s.$$

Suppose

$$\text{(15)} \qquad \begin{aligned} &\text{(a)} \quad \limsup_{s\to\infty} P(m, s) = P(m), \\ &\text{(b)} \quad \limsup_{m\to\infty} P(m) = p, \\ &\text{(c)} \quad \limsup_{m\to\infty} P(m, s) = Q(s), \\ &\text{(d)} \quad \limsup_{s\to\infty} Q(s) = q. \end{aligned}$$

Let R be the smallest of $(P(m))^{-1}$, $(m=0, 1, \cdots)$, p^{-1}, q^{-1}, and N. Then

$$\text{(16)} \qquad \limsup_{s\to\infty} A_{s+1}^{(m)}/A_s^{(m)} \leqq P(m),$$

and (14) converges for $|x| < R$. Since N is at our choice, let N be less than R. We have also

* Journal für die reine und angewandte Mathematik, vol. 76 (1873), p. 214.

$$A_{s+1}^{(m)} \leqq \prod_{i=0}^{s} P(m, i); \tag{17}$$

hence

$$\limsup_{m\to\infty} A_{s+1}^{(m)} \leqq \prod_{i=0}^{s} Q(i) = A_{s+1}$$

and

$$\limsup_{s\to\infty} A_{s+1}/A_s \leqq q.$$

Then the series

$$F(x) = \sum_{s=0}^{\infty} A_s |x|^s \tag{18}$$

converges for $|x| \leqq N < R$.

Let $M_N^{(m)}$ be such that $|h_m(x)| \leqq F_m(x) \leqq M_N^{(m)}$ and M_N such that $F(x) \leqq M_N$; then

$$\limsup_{m\to\infty} M_N^{(m)} = K_N \leqq M_N.$$

The conditions of Theorem K are satisfied and we may state the following theorem:

THEOREM. *If we have a functional equation with an operator having the property* (3), *if there exists a set of values fulfilling conditions* (8) *and* (9), *and if conditions* (15) *are satisfied, then there exists a unique expansion of the form*

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x), \tag{19}$$

where $f(x)$ is analytic for $|x| < \rho$, which will converge uniformly for $|x| \leqq R < G$, where

$$G = \min \left\{ \max_N N(1 + K_N)^{-1} \right\}.$$

The expansion converges and represents the function for $|x| < G$.

3. **Examples.** Suppose we have the equation

$$\begin{aligned} \sum_{j=0}^{n} P_{j,0}(x, \lambda)\delta^{n-j}y(x) + \sum_{i=1}^{r} \lambda^i \sum_{j=0}^{m} P_{j,i}(x, \lambda)\delta^{m-j}y(x) \\ + \int_0^x g(x, t, \lambda) y(t) dt = 0, \end{aligned} \tag{20}$$

where

$$P_{0,0}(x) \equiv 1, \quad \left| \frac{d}{dx} P_{j,i}(x, \lambda) \right| \leqq M_{N,N}^{(j,i)}, \quad |x| \leqq N < r, \quad |\lambda| > N,$$

$$g(x, t, \lambda) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij}(\lambda) x^i t^j,$$

$$G(x, p, \lambda) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij}(\lambda) \frac{i + j + 1}{i + p + 1} x^{i+j}, \quad p \text{ an integer},$$

and

$$|G(x, p, \lambda)| \leqq M_{N,N}, \qquad |x| \leqq N < r, \qquad |\lambda| > N.$$

The function $\delta^s y(x)$ is either $x^s d^s y/dx^s$, $y(q_s x)$ with $|q_n| > 1$ and $|q_n| > |q_{n-i}|$, $(i = 1, 2, \cdots, n)$, or $y(q^s x)$ with $|q| > 1$. The function $f_0(m, \lambda)$ will be a polynomial of degree r in λ and of degree n in either q^m or m, or a polynomial of degree m in q_j, $(j = 0, 1, \cdots, n)$. The conditions of the theorem can then be shown to be satisfied, and the expansion of an arbitrary function follows. Consider the case for which $r = 1$, $m = 0$, and $P_{n,i}(x, \lambda)$, $g(x, t, \lambda)$ are independent of λ. If $\{\lambda_m\}$, $(m = 0, 1, \cdots)$, is the set of characteristic values and $y_m(x)$ are the corresponding functions, then the solution of the nonhomogeneous equation

$$\text{(21)} \qquad \begin{aligned} &\delta^n y(x) + P_1(x)\delta^{n-1} y(x) + \cdots + P_0(x) y(x) \\ &\qquad + \int_0^x g(x, t) y(t) dt + \lambda y(x) = f(x), \end{aligned}$$

where $f(x)$ is analytic for $|x| < \rho$, has a solution of the form

$$\text{(22)} \qquad y(x) = \sum_{m=0}^{\infty} \frac{a_m}{\lambda - \lambda_m} y_m(x), \qquad \lambda \neq \lambda_m,$$

where $f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$. This is easily verified by substitution. If $\lambda = \lambda_p$ and $f(x) = x^{p+1} F(x)$, then the solution is of the form

$$y(x) = \sum_{m=p+1}^{\infty} \frac{a_m}{\lambda_p - \lambda_m} y_m(x).$$

Consider the equation

$$\text{(23)} \qquad \sum_{n=0}^{\infty} \frac{A_n(x, \lambda)}{n!} \left(x \frac{dy}{dx} \right)^n + \lambda y(x) = 0,$$

where

$$\frac{d}{dx}A_n(x,\lambda) \equiv 0, \quad n > n', \quad \left|\frac{d}{dx}A_n(x,\lambda)\right| \leqq M_{N,N}^{(n)},$$

$$|x| < N < r, \quad |\lambda| > N,$$

$A_n(0,\lambda) \neq 0$, $(n > n')$, $A_n(0,\lambda) = a_n$ independent of λ, and

$$\left(x\frac{dy}{dx}\right)^1 = x\frac{dy}{dx}, \quad \left(x\frac{dy}{dx}\right)^2 = x\frac{d}{dx}\left\{x\frac{dy}{dx}\right\},$$

$$\left(x\frac{dy}{dx}\right)^p = x\frac{d}{dx}\left\{\left(x\frac{dy}{dx}\right)^{p-1}\right\}.$$

Then we obtain the relations

$$f_0(m,\lambda) = \sum_{n=0}^{\infty}\frac{a_n m^n}{n!} + \lambda = 0, \quad \lambda = -\sum_{n=0}^{\infty}\frac{a_n m^n}{n!} = -f(m).$$

If $a_n = 1$, $\lambda_m = -e^m$.

This equation and others in which λ_m has the properties

$$\limsup_{m\to\infty}\frac{|P(m+s)|}{|\lambda_{m+s+1} - \lambda_m|} = Q(s),$$

$$\limsup_{s\to\infty} Q(s) = q,$$

$$\limsup_{s\to\infty}\frac{|P(m+s)|}{|\lambda_{m+s+1} - \lambda_m|} = \bar{P}(m),$$

$$\limsup_{m\to\infty} \bar{P}(m) = p,$$

$P(m+s)$ being a polynomial in $m+s$, will satisfy the conditions of the theorem, and the expansion follows.

The generalized Fuchsian equation

$$\sum_{n=0}^{\infty} x^n \frac{A_n(x,\lambda)}{n!}\frac{d^n y}{dx^n} + \lambda y(x) = 0$$

is similar to the above except for the fact that the λ_m are given by Newton series.

UNIVERSITY OF ILLINOIS