THE EQUIVALENCE OF SEQUENCE INTEGRALS AND NON-ABSOLUTELY CONVERGENT INTEGRALS*

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This note completes and extends some results previously obtained.† Let the function $f(x)$ be measurable, and finite almost everywhere on (a, b). Let $s_n(x)$ be a sequence of summable functions such that $s_n = f$ on a set E_n, $s_n = 0$ elsewhere, $E_n \supset E_{n-1}$, and $mE_n \to b-a$. If $\int_a^x s_n dx$ tends to a continuous function $\phi(x)$, then f is, by definition, *totally integrable in the sequence sense to* $\phi(x) = TS(f, a, x)$. It has been shown that if $f(x)$ is integrable in the generalized Denjoy sense to $F(x) = \int_a^x f(x)dx$, then there exists $TS(f, a, x) = F(x)$.‡ Such a function $TS(f, a, x)$ is generalized absolutely continuous (ACG),§ since $F(x)$ is (ACG). A function $TS(f, a, x)$ was constructed‖ which was not (ACG) and consequently not equal to $F(x)$. This raised the question as to whether or not the property of being (ACG) was sufficient to insure that $TS(f, a, x) = F(x)$. In the present note this question is answered in the negative, and necessary and sufficient conditions are determined for the relation $TS(f, a, x) = F(x)$.

We first construct a function $f(x)$ which is not summable, but which is integrable in a non-absolutely convergent sense, and then construct $TS(f, a, x)$ which is (ACG) and not equal to $F(x) = \int_a^x f dx$. Let G be a Cantor set on (a, b) with $mG > 0$, and let (α_i, β_i) be the intervals complementary to G. On (α_i, β_i) construct f_i such that $\int_{\alpha_i}^{\beta_i} f_i dx$ exists as a non-absolutely convergent integral with β_i the single point of non-summability of f_i, with $\int_{\alpha_i}^{\beta_i} f_i dx = 0$, and with $\left|\int_{\alpha_i}^x f_i dx\right| < \beta_i - \alpha_i$ for x on (α_i, β_i). Let $f(x) = f_i(x)$ on (α_i, β_i), and $f(x) = 0$ elsewhere. Then $F(x) = \int_a^x f dx$ exists as a non-absolutely convergent integral, and $F(x) = 0$ for x a point of G. Consider the set of intervals (α_i, β_i) ordered in any way. Then take the first n intervals of this set and order them from left to right into the set $(\alpha_1', \beta_1'), \cdots, (\alpha_n', \beta_n')$. To the right of each interval (α_j', β_j') there is an interval $\lambda_{ni} = (\beta_j', \alpha_{j+1}')$, where i is the subscript that (α_i, β_i) has in the original ordering

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† Transactions of this Society, vol. 41 (1935), pp. 171–192. In what follows this paper will be referred to as T.

‡ T, p. 186, Theorem 6.

§ Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 152, §9.

‖ T, pp. 189–191.

(α_i, β_i). On (α_j', β_j') fix a set E_{ni} such that if $s_{ni}=f$ on E_{ni} and $s_{ni}=0$ elsewhere, then

$$\left| \int_{\alpha_j'}^{\beta_j'} s_{ni}dx - m\lambda_{ni} \right| < \frac{1}{n^2}\cdot$$

If $s_n = s_{ni}$ on (α_j', β_j') and $s_n=0$ elsewhere, then

$$\left| \int_a^b s_n dx - \sum m\lambda_{ni} \right| < \frac{1}{n}\cdot$$

As n increases it is possible to choose the sets E_{ni} in such a way that $E_{(n+1)i} \supset E_{ni}$ and $mE_{ni} \to \beta_i - \alpha_i$. If this is done, it is then easily verified that

$$\lim_{n\to\infty} \int_a^x s_n dx = F(x) + mG(a, x) = TS(f, a, x),$$

where $G(a, x)$ is the part of G on (a, x). The function $F(x)$ is (ACG), and the function $mG(a, x)$ is (AC). Hence $TS(f, a, x)$ is (ACG). Furthermore, since $mG>0$, we have $TS(f, a, x) \neq F(x)$.

The foregoing considerations lead us to seek necessary and sufficient conditions that $TS(f, a, x)=F(x)$. Associated with the function $\phi(x)=TS(f, a, x)$ is a function of sets $\phi(G)=\lim \int_G s_n dx$, provided this limit exists, where G is a measurable set on (a, b) and s_n is the sequence involved in the definition of $TS(f, a, x)$. If G is an interval (α, β), then $\phi(\alpha, \beta)=TS(f, \alpha, \beta)$. The function of sets $\phi(G)$ is *completely additive* if, for every set of disjunct sets $G_1, G_2, \cdots$, we have the relation $\phi(\sum G_i)=\sum\phi(G_i)$. The function $\phi(G)$, associated with the function $f(x)$ defined above, is not completely additive. For if (α_i, β_i) is the set of open intervals complementary to the set E, then $\phi[\sum(\alpha_i, \beta_i)]=mE$, while $\sum\phi(\alpha_i, \beta_i)=0$. We prove the following theorem:

THEOREM 1. *If $f(x)$ is integrable in the generalized Denjoy sense to $F(x)$, and if $\phi(x)=TS(f, a, x)$ is such that $\phi(G)$ is completely additive, then $\phi(x)=F(x)$.*

Let E_1 be the points of non-summability of f over (a, b), (α_i, β_i) the intervals on (a, b) complementary to E_1, and (α_i', β_i') an interval with $\alpha_i<\alpha_i'<\beta_i'<\beta_i$. The function f is summable on (α_i', β_i'), and

$$\phi(\beta_i') - \phi(\alpha_i') = \int_{\alpha_i'}^{\beta_i'} f dx = F(\beta_i') - F(\alpha_i').$$

It then follows from the continuity of F and ϕ that $\phi(\beta_i)-\phi(\alpha_i)=F(\beta_i)-F(\alpha_i)$. Let E_2 be the points of non-summability of f over E_1

together with the points of E_1 at which $\sum|F(\beta_i)-F(\alpha_i)|$ diverges, let (α_j, β_j) be the intervals on (a, b) complementary to E_2, and let (α_j', β_j') be an interval with $\alpha_j<\alpha_j'<\beta_j'<\beta_j$. Then

$$\phi(\beta_j')-\phi(\alpha_j')=\lim_{n\to\infty}\int_{\alpha_j'}^{\beta_j'}s_n dx=\lim_{n\to\infty}\int_{\Sigma(\alpha_l,\beta_l)}s_n dx+\lim_{n\to\infty}\int_e s_n dx,$$

where e is the part of E_1 on (α_j', β_j'), and $\sum(\alpha_l, \beta_l)$ is the part of the set (α_i, β_i) on (α_j', β_j'). The second limit on the right exists for the reason that f is summable over e. Consequently the first limit on the right exists. Then, since $\phi(G)$ is completely additive,

$$\begin{aligned}\phi(\beta_j')-\phi(\alpha_j') &= \sum_l \lim_{n\to\infty}\int_{\alpha_l}^{\beta_l}s_n dx+\int_e f dx\\ &= \sum\{\phi(\beta_l)-\phi(\alpha_l)\}+\int_e f dx\\ &= \sum\{F(\beta_l)-F(\alpha_l)\}+\int_e f dx\\ &= F(\beta_j')-F(\alpha_j').\end{aligned}$$

Again the continuity of F and ϕ gives $\phi(\beta_j)-\phi(\alpha_j)=F(\beta_j)-F(\alpha_j)$. This process can be continued by means of finite and transfinite induction to give $\phi(x)=F(x)$ for x on (a, b).

The complete additivity of $\phi(G)$ is not a necessary condition that $\phi(x)=TS(f, a, x)=F(x)$. Let $x_0=0<x_1<x_2<\cdots$ be a sequence of values of x on $(0, 1)$ with x_n tending to unity. Let f be so defined on (x_{n-1}, x_n) that the integral of f over this interval exists as a non-absolutely convergent integral with x_n the single point of non-summability. Furthermore, let f be such that if $F(x)=\int_a^x f dx$, then $F(x_n)-F(x_{n-1})=(-1)^{n-1}n$. There then exists $TS(f, a, x)$* such that

$$\phi(x)=TS(f, a, x)=\lim_{n\to\infty}\int_0^x s_n dx=F(x).$$

The set E of points of non-summability of f is the set $x_0, x_1, \cdots$, and unity. Let (α_i, β_i) be the intervals complementary to E. We have

$$\begin{aligned}F(1)-F(0) &= \lim_{n\to\infty}\int_0^1 s_n dx=\lim_{n\to\infty}\int_E s_n dx+\lim_{n\to\infty}\int_{CE}s_n dx\\ &= \lim_{n\to\infty}\int_{\Sigma(\alpha_i,\beta_i)}s_n dx=\phi\left[\sum(\alpha_i, \beta_i)\right].\end{aligned}$$

* T, p. 186, Theorem 6.

But the value of $\sum\phi(\alpha_i, \beta_i)$ depends on the order in which the intervals (α_i, β_i) are taken.

THEOREM 2. *Let $F(x)=\int_a^x f dx$, where the integration is in the generalized Denjoy sense. Let $\phi(x)=TS(f, a, x)$. A necessary and sufficient condition that $\phi(x)=F(x)$ is that if (l, m) is an interval on (a, b) containing a closed set e over which f is summable, if (α_i, β_i) are the intervals on (l, m) contiguous to e, and if $\sum|\phi(\beta_i)-\phi(\alpha_i)|$ converges, then $\phi[\sum(\alpha_i, \beta_i)]=\sum\phi(\alpha_i, \beta_i)$.*

The proof of the sufficiency of the conditions follows the same lines as the proof of Theorem 1. To show that the conditions are necessary let $\phi(x)=TS(f, a, x)=F(x)$. Then

$$\phi(m) - \phi(l) = \lim_{n\to\infty}\int_{\Sigma(\alpha_i,\beta_i)} s_n dx + \lim_{n\to\infty}\int_e s_n dx$$

$$= \phi[\sum(\alpha_i, \beta_i)] + \int_e f dx.$$

From this, and the fact that $\phi(x)=F(x)$, it follows that $\phi[\sum(\alpha_i, \beta_i)] =\sum\{F(\beta_i)-F(\alpha_i)\}=\sum\{\phi(\beta_i)-\phi(\alpha_i)\}$.

In the foregoing f is summable over e. Suppose that f is not summable over e, but that $\int_e f dx$ exists as a non-absolutely convergent integral, and suppose that $\sum|\phi(\beta_i)-\phi(\alpha_i)|$ converges. Is it necessary that $\phi[\sum(\alpha_i, \beta_i)]=\sum\phi(\alpha_i, \beta_i)$? We answer this question in the negative. Returning to the first example above, we bisect the intervals (α_i, β_i), getting the intervals (α_i, a_i), (a_i, β_i). On each of these intervals we define f_i in the way that f_i was defined on the original interval (α_i, β_i). In particular, a_i will be the single point of non-summability of f_i on (α_i, a_i), and β_i will be the single point of non-summability of f_i on (a_i, β_i). Let $f=f_i$ on (α_i, a_i) and on (a_i, β_i), and let $f=0$ elsewhere. Going to the interval (α_j', β_j') we determine sets E_{ni}'' on (α_j', a_i) and E_{ni}'' on (a_i, β_j') in such a way that if $s_n'=f$ on E_{ni}'', and $s_n=0$ elsewhere, then $\int_a^x s_n dx \to mG(a, x)$; if $s_n''=f$ on E_{ni}'' and $s_n''=0$ elsewhere, then $\int_a^x s_n dx \to -mG(a, x)$. If further $s_n=s_n'+s_n''$, then

$$\phi(x) = \lim_{n\to\infty}\int_a^x s_n dx = TS(f, a, x) = \int_a^x f dx = F(x).$$

Let e be the closed set complementary to the set of open intervals (α_i, a_i). Then $\int_e f dx$ exists as a non-absolutely convergent integral. But $\phi[\sum(\alpha_i, a_i)]=mG$, and $\sum\phi(\alpha_i, a_i)=0$. It thus appears that in so far as it is a question of complete additivity, the conditions of Theorem 2 are the best possible.

So far it has been assumed that $f(x)$ is integrable in a non-absolutely convergent sense. We now prove the following theorem:

THEOREM 3. *Let the function $f(x)$ be measurable on (a, b), and let $TS(f, a, x)$ exist. If E is any closed set on (a, b), let an interval (l, m) containing a part e of E exist such that f is summable over e, $\sum|\phi(\alpha_i, \beta_i)|$ converges, and $\sum\phi(\alpha_i, \beta_i)=\phi[\sum(\alpha_i, \beta_i)]$, where (α_i, β_i) are the intervals complementary to e on (l, m). Then f is integrable in the generalized Denjoy sense, and $\int_a^x f dx = TS(f, a, x)$.*

If the closed set E of the theorem is the interval (a, b), then the part e of E on (l, m) is all of (l, m), and it follows that the points E_1 of non-summability of f over (a, b) are non-dense on (a, b). Let (α, β) be an interval complementary to E_1. The function f is then summable over any interval interior to (α, β), and as a consequence of this $\phi' = f$ almost everywhere on (α, β). The set E_1 is closed. By the conditions of the theorem there is an interval (l, m) containing a part e of E_1 with f summable over e, $\sum|\phi(\alpha_i, \beta_i)|$ convergent, and $\sum\phi(\alpha_i, \beta_i) = \phi[\sum(\alpha_i, \beta_i)]$. For x on this interval (l, m), we have

$$\begin{aligned}\phi(x) - \phi(l) &= \lim_{n\to\infty}\int_l^x s_n dx \\ &= \lim_{n\to\infty}\int_{e(l,x)} s_n dx + \lim_{n\to\infty}\int_{\Sigma_{(l,x)}(\alpha_i,\beta_i)} s_n dx \\ &= \int_e f dx + \sum_{(l,x)}\phi(\alpha_i, \beta_i) + \phi(\alpha_k, x),\end{aligned}$$

where the second term on the right represents the whole intervals of the set (α_i, β_i) on (l, x), and the third term is absent unless x is an interior point of an interval (α_k, β_k) of the set (α_i, β_i). Set $\psi(x) = \phi_1(x) + \phi_2(x)$, where

$$\phi_1 = \int_{e(l,x)} f dx, \qquad \phi_2 = \sum_{(l,x)}\phi(\alpha_i, \beta_i).$$

The function ψ is constant on (α_i, β_i) and equal to ϕ at points of e. Furthermore, $\phi_1' = f$ at almost all of e, and $\phi_2' = 0$ at almost all of e.* Consequently $\psi' = f$ almost everywhere on e, and since $\psi = \phi$ at points of e, it follows that ϕ has an approximate derivative equal to f at almost all points of e. If now E_2 denotes the points of E_1 which are points of non-summability of f over E, or the points of E_1 at

* Denjoy, Journal de Mathématique, (7), vol. 1, p. 158 ff.

which $\sum|\phi(\alpha_i, \beta_i)|$ either diverges or converges with $\sum\phi(\alpha_i, \beta_i) \neq\phi[\sum(\alpha_i, \beta_i)]$, then the conditions of the theorem and the above reasoning allow us to conclude that the set E_2 is non-dense on E_1, and that if (α, β) is an interval of the set complementary to E_2, then ϕ has an approximate derivative equal to f at almost all points of this interval. This process can be continued by finite and transfinite induction to show that ϕ has an approximate derivative equal to f almost everywhere on (a, b).

It will next be shown that $\phi(x) = TS(f, a, x)$ is (ACG). It is sufficient to show that for every closed set E there exists an interval containing a part e of E over which ϕ is absolutely continuous.* Let E be any closed set on (a, b), and (l, m) an interval containing a part e of E over which f is summable and for which $\sum|\phi(\alpha_i, \beta_i)|$ converges with $\sum\phi(\alpha_i, \beta_i) = \phi[\sum(\alpha_i, \beta_i)]$. Let (a', b') be an interval on (l, m) with a', b' points of e. Then

$$|\phi(b') - \phi(a')| = \left|\int_{e(a',b')} fdx + \sum_{(a',b')} \phi(\alpha_i, \beta_i)\right|.$$

Since $\sum\phi(\alpha_i, \beta_i)$ converges, it is clear that the right-hand side of this equation can be made arbitrarily small by taking $b' - a'$ sufficiently small. Similarily, $\sum|\phi(b_k') - \phi(a_k')|$ is arbitrarily small if (a_k', b_k') is a finite or denumerably infinite set of intervals of (l, m) with a_k', b_k' points of e and $\sum(b_k' - a_k')$ sufficiently small. But this means that ϕ is (AC) on e and consequently (ACG) on (a, b).

We now have $\phi(x)$ (ACG) on (a, b) and the approximate derivative of ϕ equal to f almost everywhere. This allows us to conclude that f is integrable in the generalized Denjoy sense,† and that

$$\phi(x) = \int_a^x fdx = TS(f, a, x).$$

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* Saks, loc. cit., p. 165.

† Saks, loc. cit., p. 197, §2.