ON COMPLETELY CONTINUOUS LINEAR TRANSFORMATIONS*

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We shall deal with complete linear vector or so-called Banach spaces.† A completely continuous linear transformation is defined as a linear transformation which carries every bounded set into a compact set. In spaces of a finite number of dimensions, that is, spaces which are linear closed extensions of a finite number of elements, all bounded sets are compact.‡ Therefore singular transformations, that is to say, linear limited transformations which transform their domains into spaces of a finite number of dimensions, are completely continuous linear transformations. It is well known that the strong limit, or limit in the norm sense, of a sequence of completely continuous linear transformations is also completely continuous and linear.§ Consequently, the strong limit of a sequence of singular transformations is completely continuous and linear. The question naturally arises whether, conversely, every completely continuous linear transformation is the strong limit of a sequence of singular transformations.‖ This paper obtains a result for the domain of the transformation, a Banach space, and the range, a space to be defined and hereafter to be referred to as of type A. It will be seen that the conception of a space of type A is really a generalization of the idea of a Banach space with a denumerable base,¶ which will hereafter be referred to as of type S.

By a space of type A we shall mean a Banach space in which there exists a linearly independent sequence $\{f_n\}$ of elements of unit norm and a double sequence $\{L_{mn}(g)\}$ of linear limited operators such that for every g

$$\lim_{m=\infty} \left\| g - \sum_{n=1}^{m_n} L_{mn}(g) f_n \right\| = 0. \tag{1}$$

* Presented to the Society, September 10, 1937.

† Banach, *Théorie des Opérations Linéaires*, p. 53.

‡ Riesz, Acta Mathematica, vol. 41 (1927), p. 77.

§ Banach, loc. cit., p. 96.

‖ Hildebrandt, this Bulletin, vol. 37 (1931), p. 196.

¶ By a space of type S we shall mean a Banach space with a finite or denumerably infinite set of elements $\{f_i\}$ of unit norm such that every element g may be uniquely represented in the form $g=\sum_{i=1}^{\infty} c_i(g) f_i$, or $\lim_{n=\infty}\|g-\sum_{i=1}^{n} c_i(g) f_i\|=0$, where for a fixed index i the coefficients $c_i(g)$ are bounded linear operators on the space. See Schauder, Mathematische Zeitschrift, vol. 26 (1927), p. 47, and Banach, loc. cit., p. 110.

It is not required that there be only one double sequence $\{L_{mn}\}$ satisfying relation (1) for a given space.

Obviously a space of type S is also of type A. It is interesting to note that the set of spaces of type A also includes as a subset spaces which have an integral representation instead of the series representation of the spaces of type S. We shall designate such spaces by E and define them as follows. A space of type E is a Banach space in which there is a set of elements $f(t)$, $(0 \leqq t \leqq 1)$, of the power of the continuum, each of unit norm, such that every element g of E is of the form

$$g = \int_0^1 f(t) d_t \lambda(g, t),$$

or

$$\lim_{N_\sigma \to 0} \left\| g - \sum_\sigma f(\tau_i)\{\lambda(g, t_i) - \lambda(g, t_{i-1})\} \right\|$$

$$= \lim_{N_\sigma \to 0} \left\| g - \sum_\sigma f(\tau_i)\Delta_i\lambda(g) \right\| = 0,$$

where the points $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ are elements of a partition σ of the interval (0, 1) and $t_{i-1} \leqq \tau_i \leqq t_i$. For each value of t, $\lambda(g, t)$ will be a bounded linear operator on the space, whence the same may be said of $\Delta_i\lambda(g)$ for each value of i. It will be understood that the integral is taken in the Riemann sense, and we shall mean by the norm of σ, written N_σ, the maximum of the lengths of the intervals formed by the partition σ.

To show that a space E is of type A it is only necessary to choose a sequence of subdivisions $\{\sigma_n\}$ such that $N_{\sigma_n} \leqq 1/n$ and a sequence of points τ_{in}, $(i = 1, 2, \cdots, n)$, such that

$$\lim_{N_{\sigma_n} \to 0} \left\| g - \sum_{\sigma_n} f(\tau_{in})\Delta_{in}\lambda(g) \right\| = 0,$$

for every g, where the second subscript after both τ and Δ refers to the partition σ_n.

It may easily be shown that every space of type S is of type E as well as of type A.

Because the spaces of type A include those of types S and E as special cases, we shall confine our attention to the former.

For each value of m, $T_m(g) = g - \sum_{n=1}^{m_n} L_{mn}(g) f_n$ is a linear limited transformation on a space A of type A. This follows directly from the linear limitedness of each $L_{mn}(g)$. We then have this lemma:

LEMMA. *The sequence* $\{T_m(g)\}$ *of linear limited transformations is such that* $\lim_{m=\infty}\|T_m(g)\|=0$ *uniformly on every self-compact partial set* H *of* A.

From the linear limitedness of each $T_m(g)$ and the fact that $\lim_{m=\infty}\|T_m(g)\|=0$ for each g of A, it follows that there exists an $M>0$, independent of both g and m, such that $\|T_m(g)\|<M\|g\|$.* By the total boundedness of the self-compact set H of E,† if $e>0$ is arbitrarily chosen then there is a finite set $g_1, g_2, \cdots, g_p$ of H such that every g of H is interior to at least one of the spheres of centers g_i, $(i=1, 2, \cdots, p)$, and of radius e/M. Obviously $\|T_m(g)\|$ approaches zero uniformly on the finite set g_i, $(i=1, 2, \cdots, p)$, so that for $n\geqq m(e)$ we have $\|T_n(g)\|\leqq e$ on this set. Then for any g of H and some g_i,

$$\big|\,\|T_m(g)\| - \|T_m(g_i)\|\,\big| \leqq \|T_m(g - g_i)\| \leqq M\|g - g_i\| \leqq e,$$

whence $\|T_n(g)\|\leqq e$ for each g of H when $n\geqq m(e)$. This proves our lemma.

THEOREM. *Every completely continuous linear transformation of a Banach space into a space of type* A *is the strong limit of a sequence of singular transformations.*

Let U be a completely continuous linear transformation of a Banach space D to a space A of type A with the base $\{f_n\}$, and let $g=U(x)$, where x is of D. Then $U_m(x)=\sum_{n=1}^{m_n}L_{mn}(g)f_n$ is a singular transformation of D into the linear closed extension of the finite number of elements f_n, $(n=1, 2, \cdots, m_n)$, which forms a subset of A.

Let A' be the transform by U of those elements D' of D whose norms are less than or equal to unity. Since U is completely continuous the set A' is self-compact, and by the previous lemma, $\|g-\sum_{n=1}^{m_n}L_{mn}(g)f_n\|$ approaches zero uniformly on A', consequently its equal $\|U(x)-U_m(x)\|$ must do likewise on D'. Hence the norm of the difference between the completely continuous linear transformation U and the singular transformation U_m approaches zero with $1/m$. This completes the proof.

From the above theorem and the observations of the introduction it follows directly that a linear transformation of a Banach space to a space of type A is completely continuous if and only if it is the strong limit of a sequence of singular transformations.

Since Hilbert space, the space of all continuous functions on a finite

* Banach, loc. cit., p. 80, Theorem 5.

† Hahn, *Reelle Funktionen*, 1921, p. 89.

closed interval with norm the absolute value of the function, and the space of all functions which are Lebesgue integrable to the pth power, $p\geqq 1$, with norm the pth root of the integral of the pth power of the absolute value of the function, are all spaces with a denumerable base in the sense of Schauder and Banach, and consequently of type A, the above theorem holds of all completely continuous linear transformations with Banach spaces as domains and such spaces as ranges.*

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MULTIVALENT FUNCTIONS OF ORDER p†

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1. **Introduction.** For the class of k-wise symmetric functions

(1.1) $$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad a_1 = 1, \ a_n = 0 \text{ for } n \not\equiv 1 \pmod{k},$$

which are regular and univalent within the unit circle, it has been conjectured that there exists a constant $A(k)$ so that for all n

(1.2) $$|a_n| \leqq A(k) n^{2/k-1}.$$

Proofs of this inequality for $k=1, 2, 2, 3$, were given by J. E. Littlewood,§ R. E. A. C. Paley and J. E. Littlewood,|| E. Landau,¶ and V. Levin** respectively. As far as the author is aware there is no valid proof†† for $k>3$ in the literature as yet.

It is the purpose of this note to point out that the methods of proof

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* Hildebrandt, this Bulletin, vol. 36 (1931), p. 197.

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‡ The author is indebted to the referee for helpful suggestions which led to a revision of this note.

§ See J. E. Littlewood, *On inequalities in the theory of functions*, Proceedings of the London Mathematical Society, (2), vol. 23 (1925), pp. 481–519.

|| See R. E. A. C. Paley and J. E. Littlewood, *A proof that an odd schlicht function has bounded coefficients*, Journal of the London Mathematical Society, vol. 7 (1932), pp. 167–169.

¶ See E. Landau, *Über ungerade schlichte Funktionen*, Mathematische Zeitschrift, vol. 37 (1933), pp. 33–35.

** See V. Levin, *Ein Beitrag zum Koeffizientproblem der schlichten Funktionen*, Mathematische Zeitscrift, vol. 38 (1934), pp. 306–311.

†† See K. Joh and S. Takahashi, *Ein Beweis für Szegösche Vermutung über schlichte Potenzreihen*, Proceedings of the Imperial Academy of Japan, vol. 10 (1934) pp. 137–139. The proof therein was found to be defective: see Zentralblatt für Mathematik, vol. 9 (1934), pp. 75–76.